Category Theory

An abstract theory of functional programming

Hype for Types

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31 March 2020

Section 1

Motivation

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But what does it mean for SML to "have" a certain type?

```
type 'a list = 'a * bool
```

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We think of the type system as a mathematical object in its own right,consisting of

- Types
- Arrows between those types: total functions

A Bird's-Eye View of SML

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int list
•

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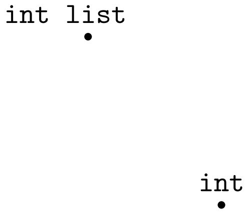

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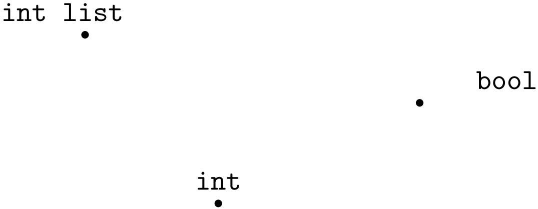

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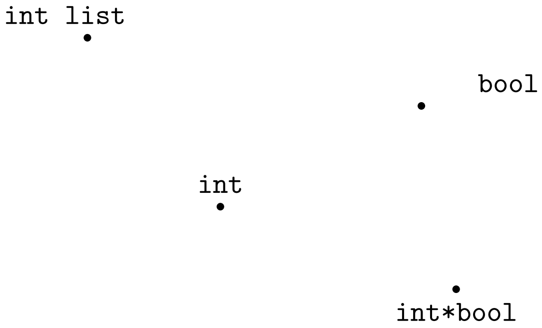

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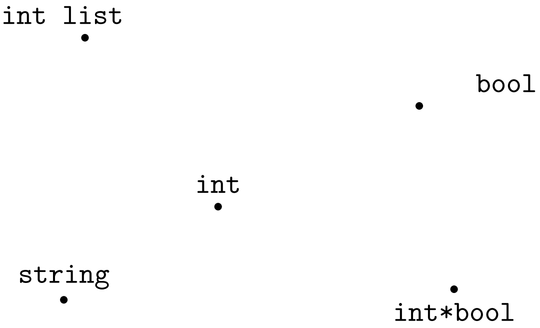

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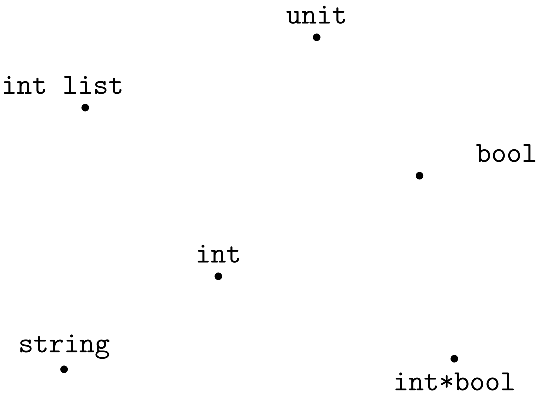

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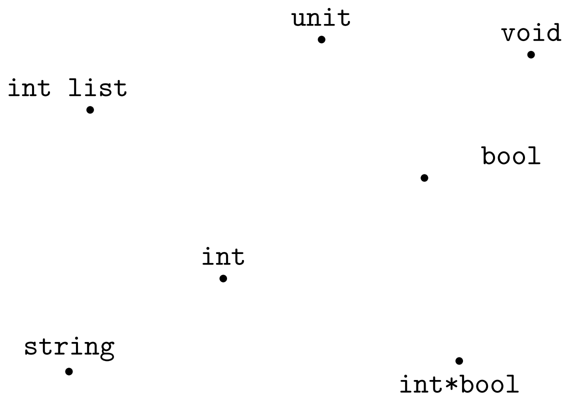

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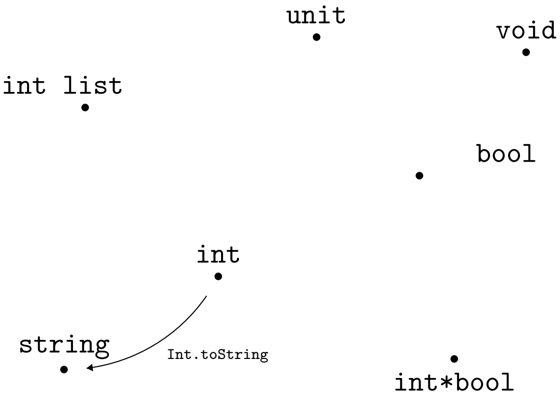

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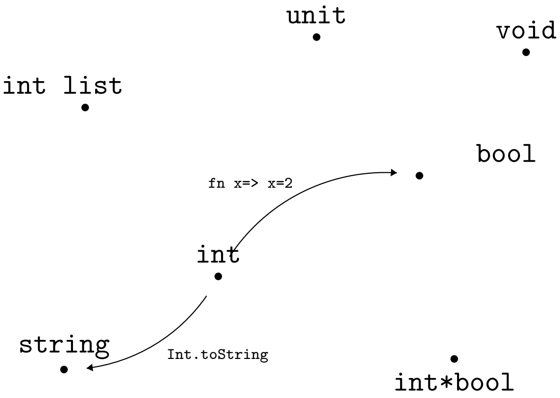

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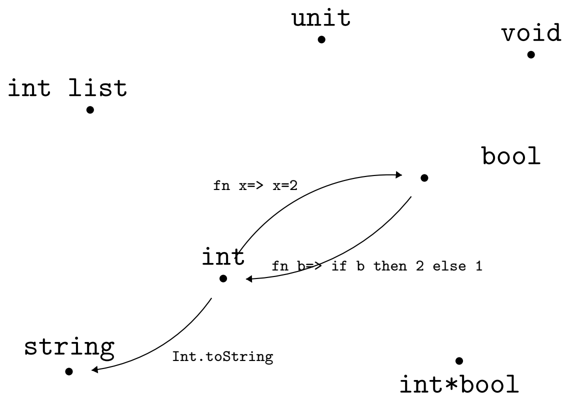

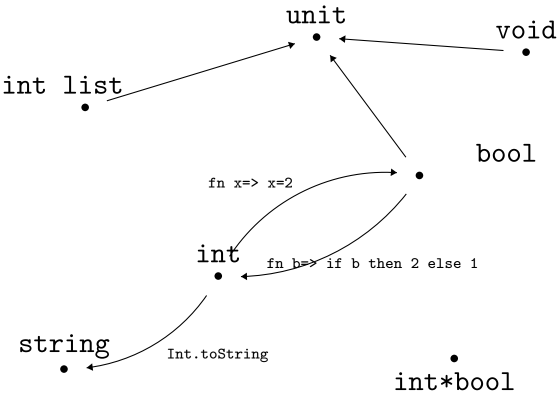
unit
void
int list
bool
fn x=> x=2
int
fn b=> if b then 2 else 1
string
Int.toString
int*bool

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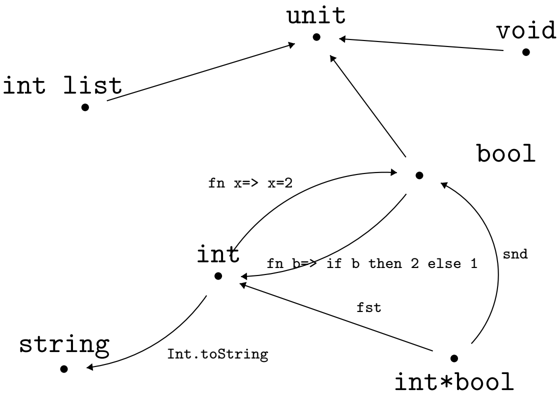

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For all types τ, *there exists a unique function* $\mathtt{u}_\tau : \tau \to \mathtt{unit}$

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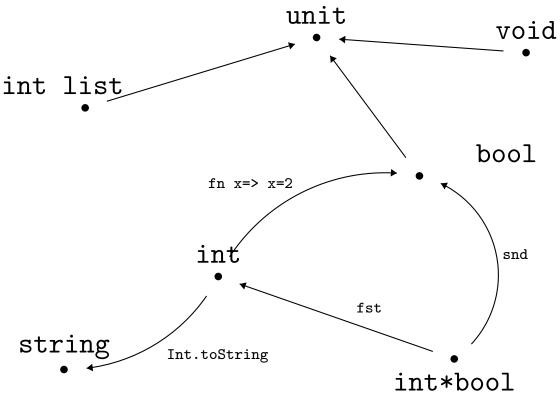

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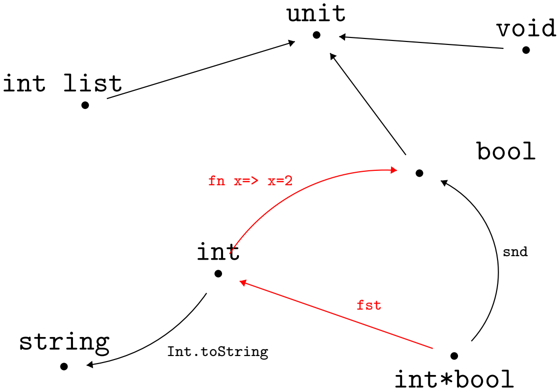

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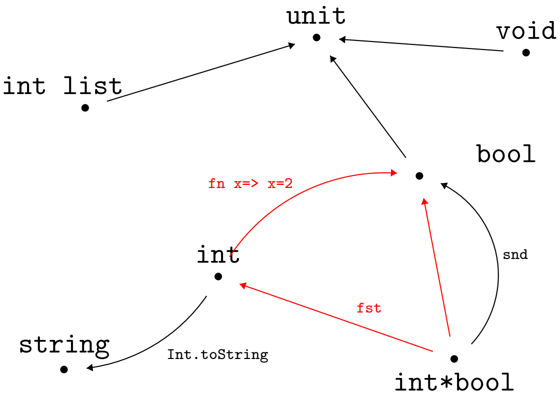

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The theory of `op o` is called **category theory**.

Section 2

Categories

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4. Composition is associative: for all $f : A \to B$, $g : B \to C$, $h : C \to D$,

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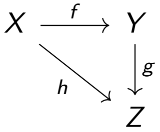

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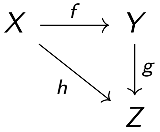

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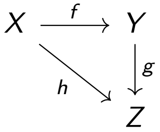

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Section 3

Universal Mapping Properties

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An object which has these properties is called a *terminal object*.

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Yes! The type `datatype void = Void of void` is initial, because, given any other type τ, the function

$$\texttt{(fn \_ => raise Fail "Won't happen")} : \texttt{void} \to \tau$$

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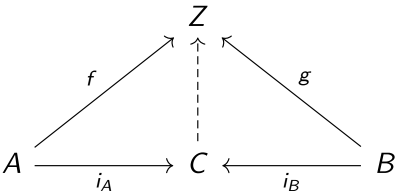

For every object Z and arrows $f : A \to Z$ and $g : B \to Z$, there exists a unique arrow $h : C \to Z$ such that

$$f = h \circ i_A \qquad \text{and} \qquad g = h \circ i_B$$

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and for any type ρ and any `f : ` $\tau \to \rho$, `g : ` $\sigma \to \rho$,

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You can check: `h o inL` $=$ `f` and `h o inR` $=$ `g`.

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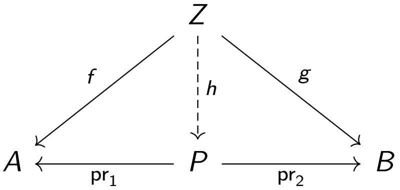

For every object Z and arrows $f : Z \to A$ and $g : Z \to B$, there exists a unique arrow $h : Z \to C$ such that

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So then...

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sectionpage

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Mathematics studies *transformations* between objects

Mathematical fields are generally concerned with certain kinds of objects and functions between them:

- Group Theory: Groups and group homomorphisms
- Linear Algebra: Vector spaces and linear transformations
- Topology: Topological spaces and continuous functions between them
- Category Theory: Categories and ...FUNCTORS!

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In SML, we already have a notion called *functors*, which are things that map between *structures*.

This usage is related, but not the same. We'll use *functor* to mean a kind of "function between categories". In this lecture, we'll focus on "endofunctors": functors from the SML type system to itself.

Defn: An *endofunctor* `F` on the SML type system consists of

- A polymorphic type constructor `'a F.t`
- A polymorphic function

```
F.map :  ('a -> 'b) -> 'a F.t -> 'b F.t
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such that, for all `f:t1 -> t2` and all `g:t2 -> t3`,

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F.map (g o f) = (F.map g) o (F.map f)
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We think of `F.t` as being a *function on types*, the type-level component of `F`. We think of `F.map` as being a *function on functions*, the function component of `F`.

Examples

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  Check that:

```
map (g o f) (x,z) = (g(f(x)),z)
                  = map g (f(x),z)
                  = map g (map f (x,z))
```

Examples

- If `F` and `G` are endpfunctors, then `G o F` is an endofunctor with `'a (G o F).t = 'a F.t G.t`
- For some type `t1`, let `'a F.t = t1 -> 'a`, we need

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- Can we define an endofunctor `F` with `'a F.t = 'a -> t1`?
  Answer: No. If we did, we would need `F.map` to be of type

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Section 5

Natural Transformations

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- Consider arrows between functors?

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- A polymorphic function `E : 'a F.t -> 'a G.t`.

such that for all functions `f : t1 -> t2`,

$$\mathtt{E_{t2}\ o\ (F.map\ f)} = \mathtt{(G.map\ f)\ o\ E_{t1}}$$

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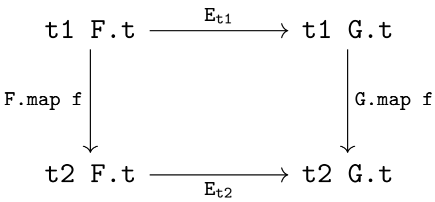
$$\mathtt{E}_{\mathtt{t2}} \ \mathtt{o} \ (\mathtt{F.map\ f}) = (\mathtt{G.map\ f}) \ \mathtt{o} \ \mathtt{E}_{\mathtt{t1}}$$



We write $\mathtt{E}_{\mathtt{t1}}$ to denote `E`, instantiated at type `t1`, i.e. $\mathtt{E}_{\mathtt{t1}}$ `: t1 F.t -> t1 G.t`.

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- The function `SOME : 'a -> 'a option` is a natural transformation from the identity functor `Id` (`'a Id.t = 'a` and `Id.map f = f`) to the `option` endofunctor.

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- The function `hd: 'a list -> 'a option` is a natural transformation from the `list` endofunctor to the `option` endofunctor
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- The function `SOME : 'a -> 'a option` is a natural transformation from the identity functor `Id` (`'a Id.t = 'a` and `Id.map f = f`) to the `option` endofunctor.
- For any endofunctor `F`, the identity function `I: 'a F.t -> 'a F.t` given by `I(x)=x` is a natural transformation `F` $\to$ `F`.

The category of endofunctors

We can form a category `Fun(SML,SML)` of endofunctors and natural transformations, where

- The objects are endofunctors on the SML type system
- The arrows are natural transformations

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We can check that the composition of endofunctors defined earlier is associative, that the identity transformation works as an identity arrow, etc.

Monoid in the category of endofunctors

Defn: An endofunctor `T` is said to constitute a *monad* if it comes equipped with two natural transformations

- `eta` : `Id` $\rightarrow$ `T`
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- $\texttt{eta} : \texttt{Id} \to \texttt{T}$
- $\texttt{mu} : \texttt{T o T} \to \texttt{T}$

such that, for all types τ,

- "`mu` is associative":

$$\texttt{mu}_\tau \texttt{ o (T.map mu}_\tau\texttt{)} = \texttt{mu}_\tau \texttt{ o mu}_{(\tau \texttt{ T.t})}$$

- "`eta` is a unit for `mu`":

$$\texttt{mu}_\tau \texttt{ o eta}_{(\tau \texttt{ T.t})} = \texttt{mu}_\tau \texttt{ o T.map eta}_\tau$$

Example

- `list` is a monad. `eta` is the singleton function

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fun eta (x : 'a) : 'a list = [x]
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and `mu` is `concat`:

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- Options are monads
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- The functor `'a F.t = ('a -> void) -> void` is a monad

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- Other fun stuff?

Thank you!