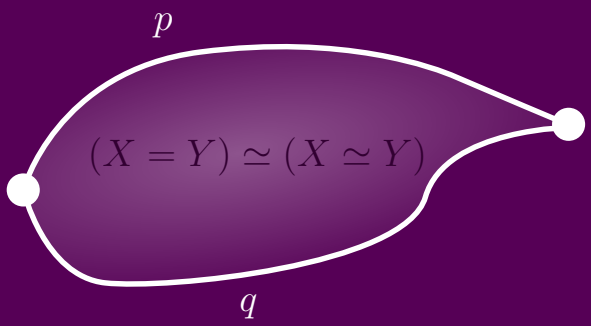


HoTT in 10 Minutes

Type Theory with a Vitamin K Deficiency

Jacob Neumann

98-317 Guest Lecture - 04 May 2021

Learning HoTT this fast is hard!

More resources for learning:

- Join our Homotopy Type Theory reading group!
- HoTT book: homotopytypetheory.org/book/
- Intro to HoTT: github.com/EgbertRijke/HoTT-Intro
- My Youtube videos: bit.ly/3eR4xkR

What HoTT's got

- Unit type, $\mathbb{1}$
- Empty type, $\mathbb{0}$
- Boolean type, $\mathbb{2}$
- Product types, sum types, function types, ...
- Inductive types ($\mathbb{N}$, Lists, Trees, etc.)
- Dependent Types

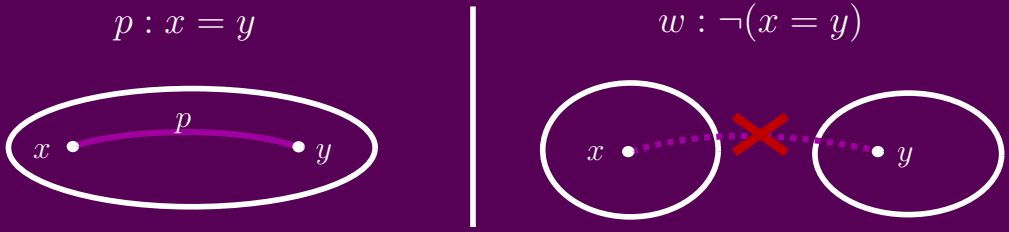
Identity Types

$$\frac{x : A \qquad y : A}{(x =_A y)\ \mathsf{type}}$$

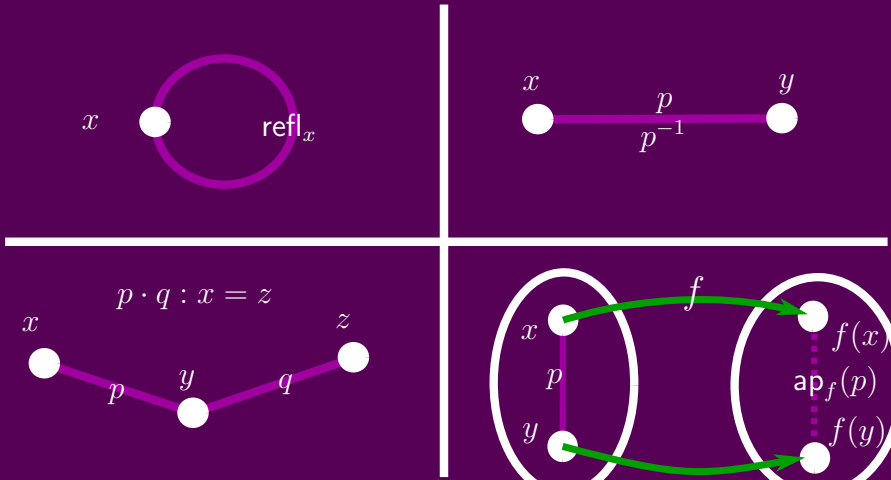
```
data Id {A : Type} (x : A) : A → Type where
    refl : Id x x
```

Curry-Howard

$p : x =_A y$ is a "proof" or "witness" of the fact that $x =_A y$.



Basic Properties of Identity Types



Axiom K

What about identities between identities?

```
K : {X : Type} → (x : X) → (p : Id x x) → Id
  p refl
```

```
UIP : {X : Type} → (x y : X) → (p q : Id x y)
   → Id p q
```

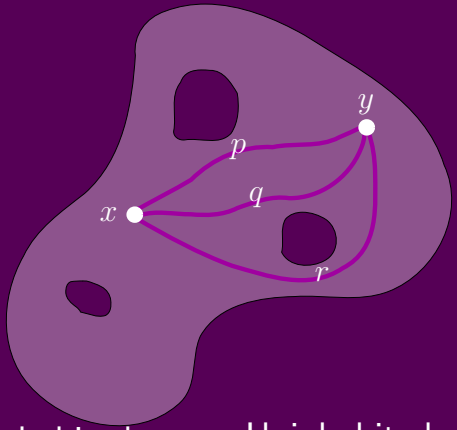
Starting-point of HoTT: Reject K

- In HoTT, we do not assume that K/UIP holds in general (though it does for many types, like $\mathbb{2}$)

```
{-# OPTIONS --without-K #-}
```

- Much of the research in HoTT is into "higher inductive types" (HITs), which are inductively-given types which have constructors for building non-refl identities

Why?



Inhabited:

$x = y$
$p = q$

Uninhabited:

$p = r$
$q = r$

An example of a HIT: the circle

Classical definition:

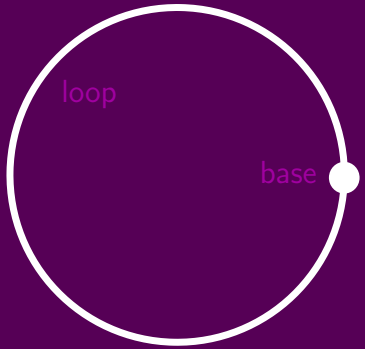
$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$$

HoTT definition:

- base : $\mathbb{S}^1$
- loop : base $=$ base

Inhabited: base $=$ base, loop $=$ loop, loop $\cdot$ loop$^{-1} = \mathsf{refl}_{\mathsf{base}}$, ...

Uninhabited: loop $= \mathsf{refl}_{\mathsf{base}}$, loop $=$ loop^{-1}, loop $=$ loop $\cdot$ loop,...



Greatest HITs

- $\mathbb{S}^2$ is a hollow sphere
  - N : $\mathbb{S}^2$
  - S : $\mathbb{S}^2$
  - merid : $\mathbb{S}^1 \to (N = S)$
- $\mathbb{I}$ is the interval
  - $0 : \mathbb{I}$
  - $1 : \mathbb{I}$
  - i : $0 = 1$
- T^2 is a hollow torus: $\mathbb{S}^1 \times \mathbb{S}^1$

Thank you!

Email me at jacobneu@andrew.cmu.edu

if you want to learn more HoTT!