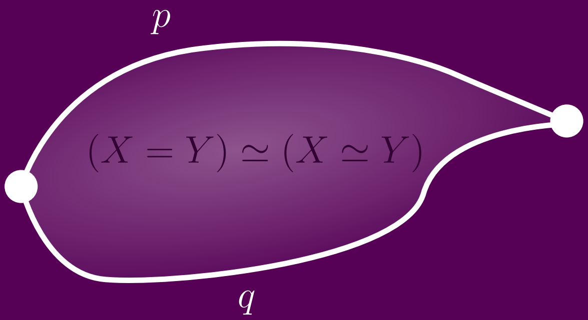


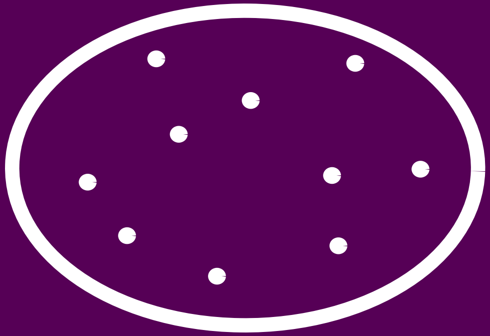
HoTT in 10 Minutes

Type Theory with a Vitamin K Deficiency

Jacob Neumann

98-317 Guest Lecture - 08 December 2020

Start with Martin-Löf Type Theory



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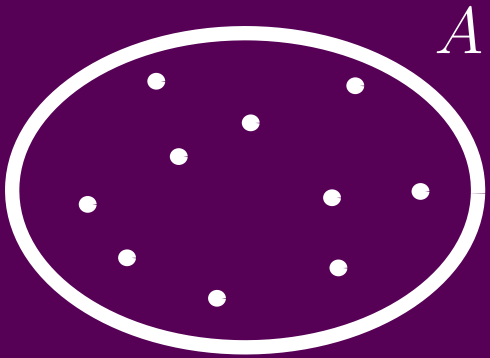

A : type

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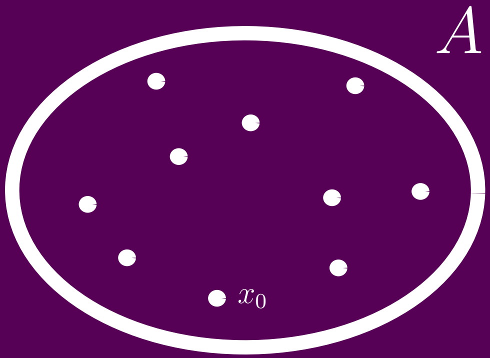

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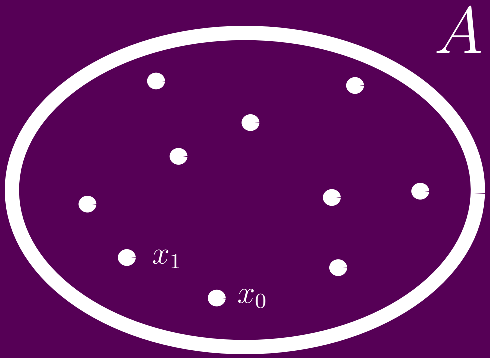

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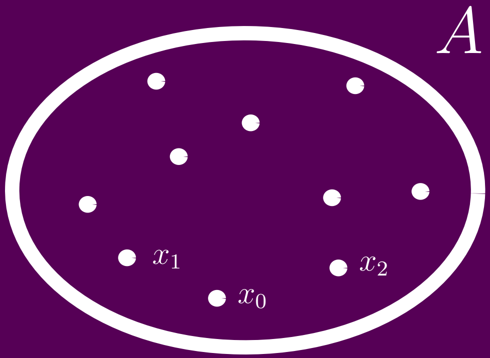

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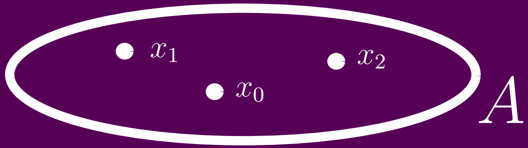

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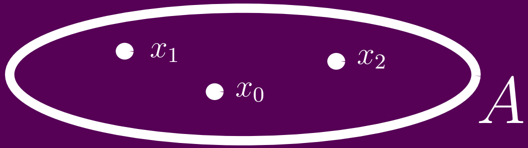

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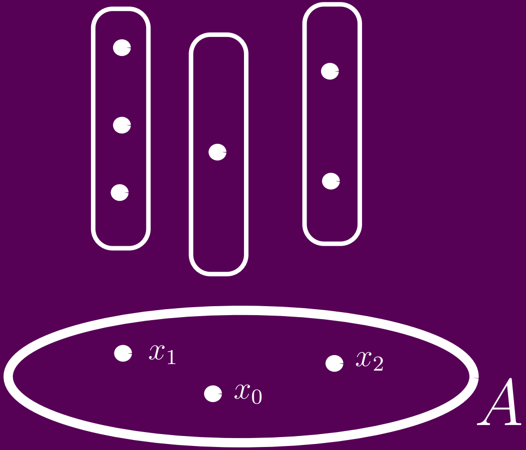

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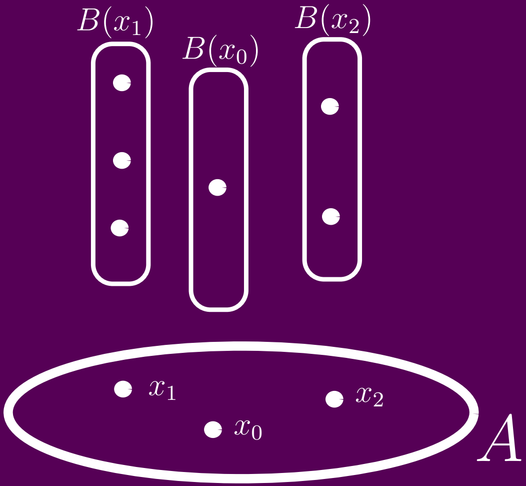

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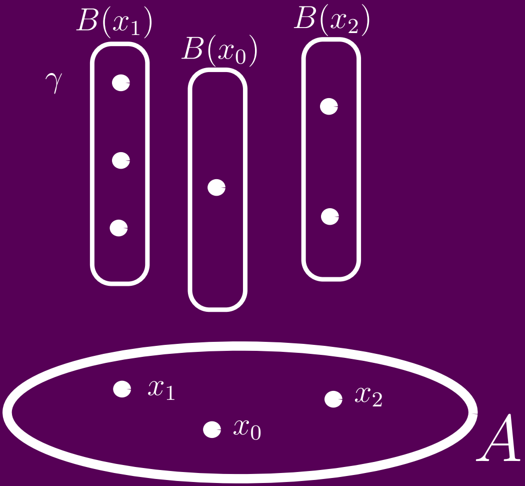

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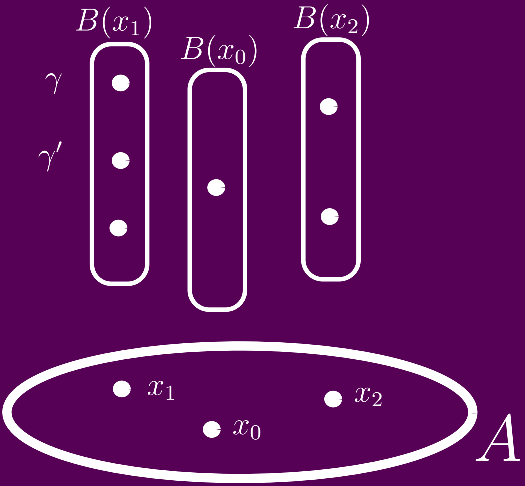
$\gamma : B(x_1)$



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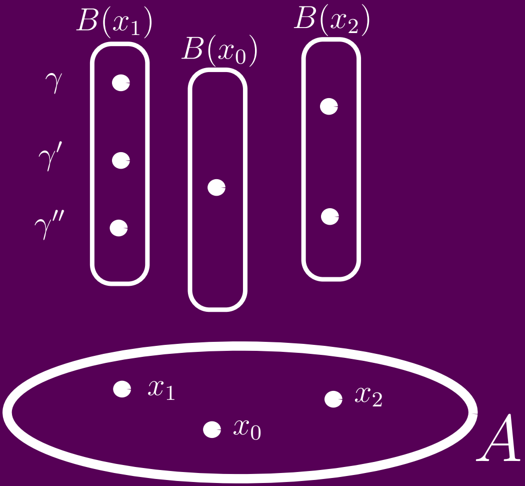

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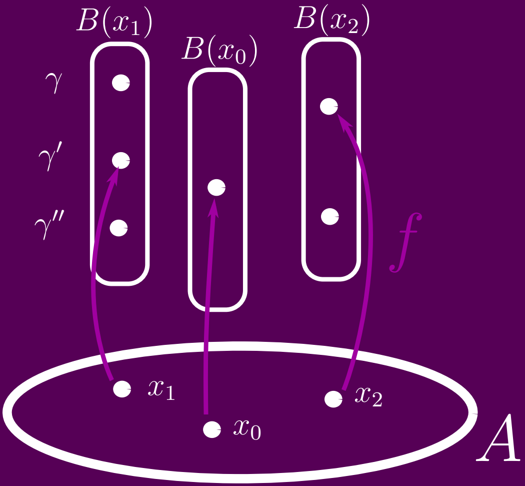

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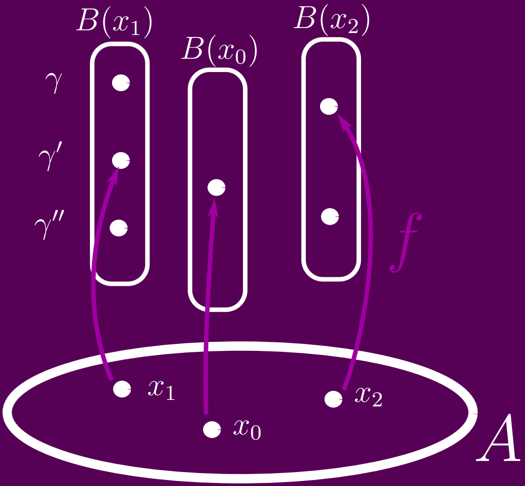

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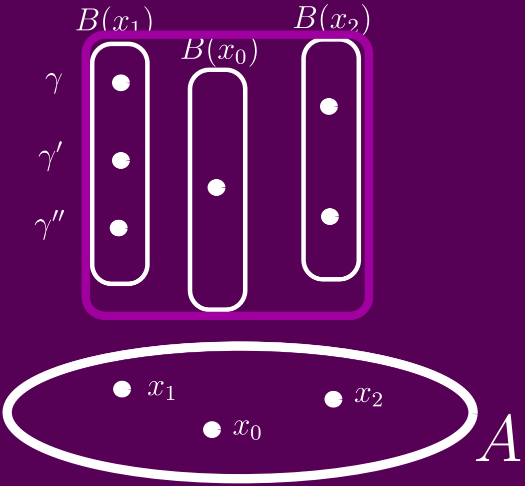

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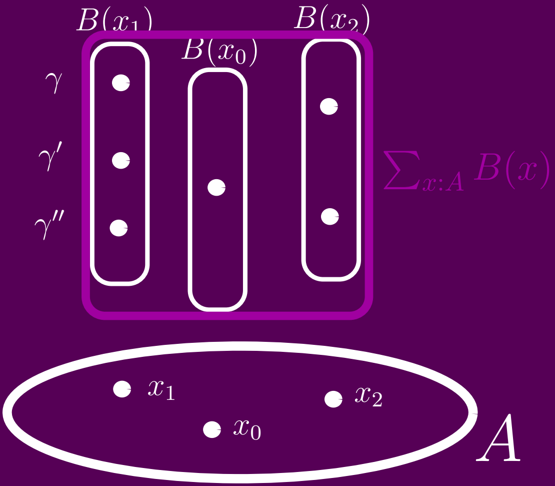

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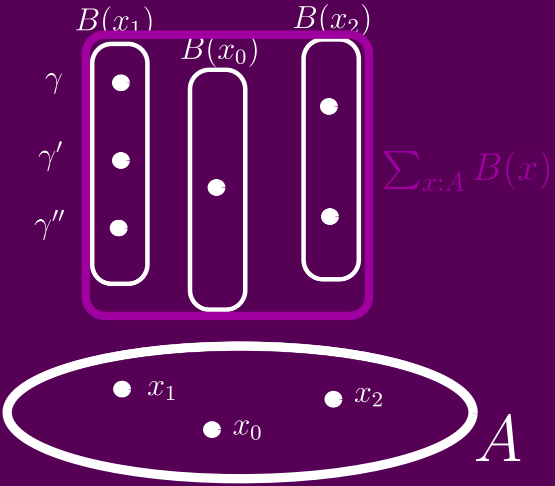

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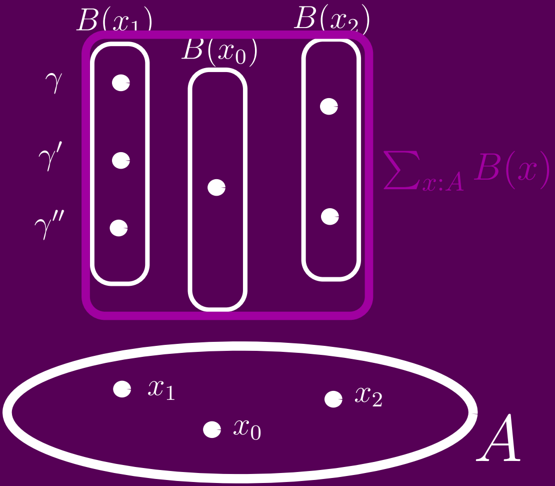
$(x_1, \gamma) : \sum_{x:A} B(x)$



$$A : \text{type}$$

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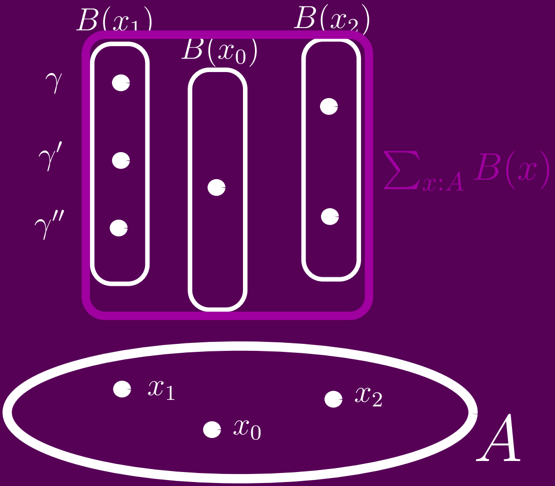

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$(x_1, \gamma) : \sum_{x:A} B(x)$

$(x_1, \gamma') : \sum_{x:A} B(x)$

$(x_1, \gamma'') : \sum_{x:A} B(x)$

Also in HoTT

- Unit type, $\mathbb{1}$
- Empty type, $\mathbb{0}$
- $\neg A \equiv (A \to \mathbb{0})$
- Boolean type, $\mathbb{2}$
- Natural numbers type, $\mathbb{N}$
- Integers type, $\mathbb{Z}$
- Product types, sum types, function types, . . .
- Inductive types

Identity Types

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$$\frac{x : A \qquad y : A}{(x = y)\ \text{type}}$$

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Curry-Howard: $p : x = y$ is a “proof” or “witness” of the fact that $x = y$.

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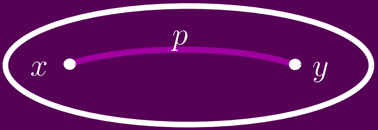
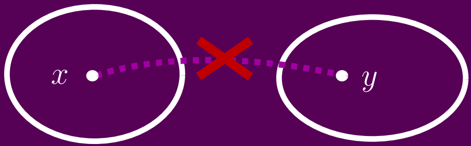
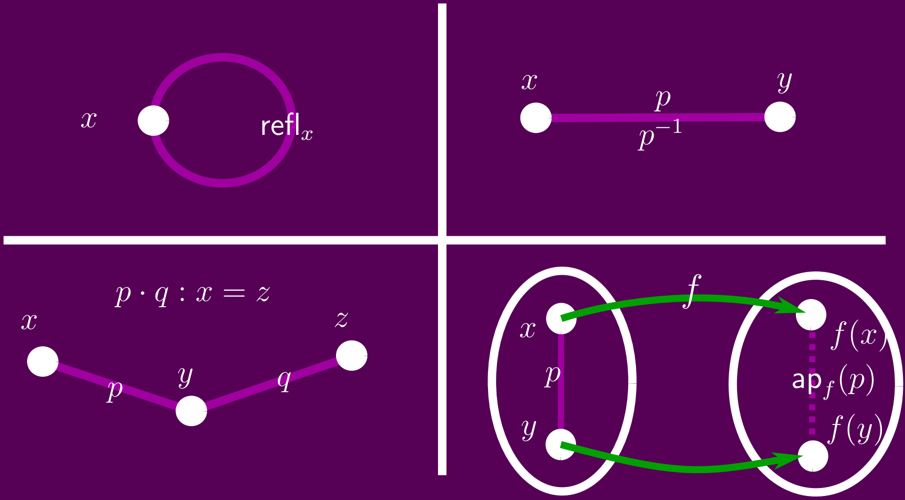
$p : x = y$



$w : \neg(x = y)$



Basic Properties of Identity Types



Axiom K

What about identities between identities?

Axiom K

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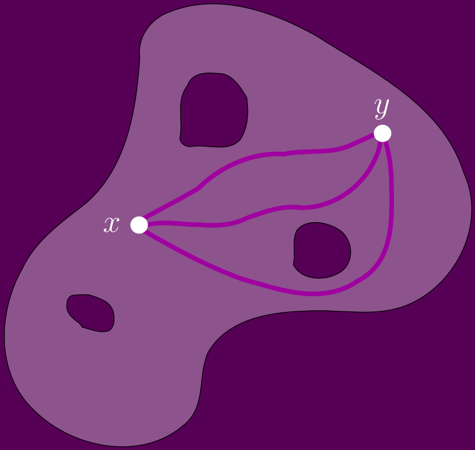

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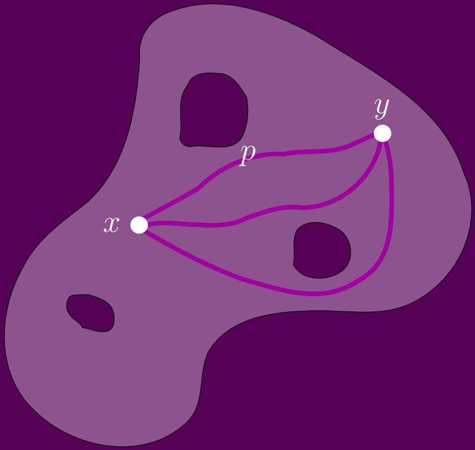

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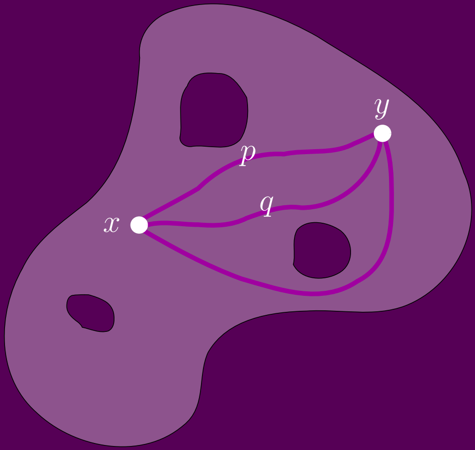

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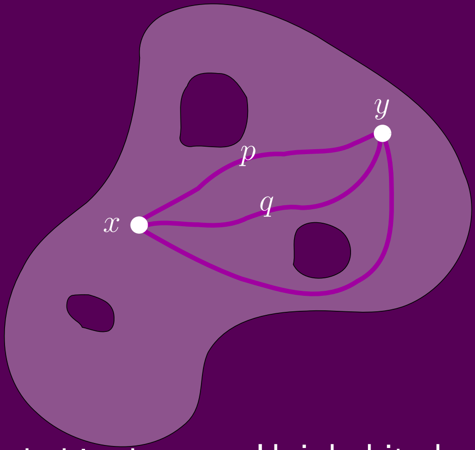

Inhabited:

Uninhabited:

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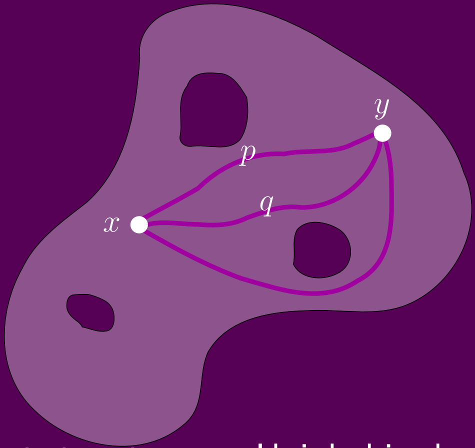

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$x = y$

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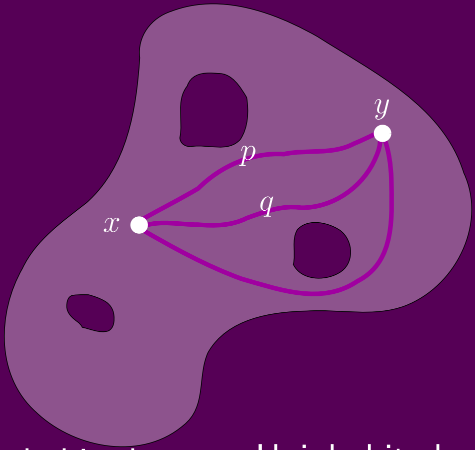

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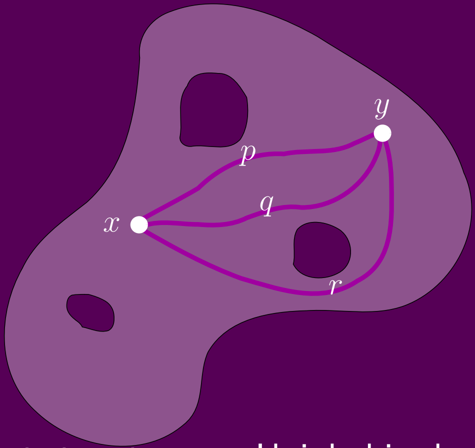

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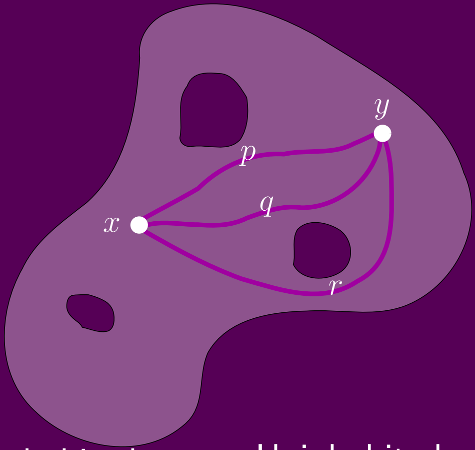

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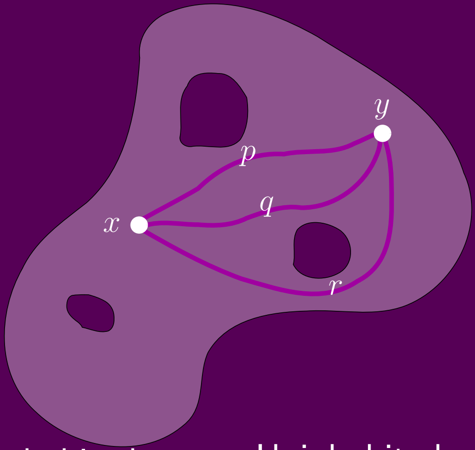
$\mathsf{K}_X :$



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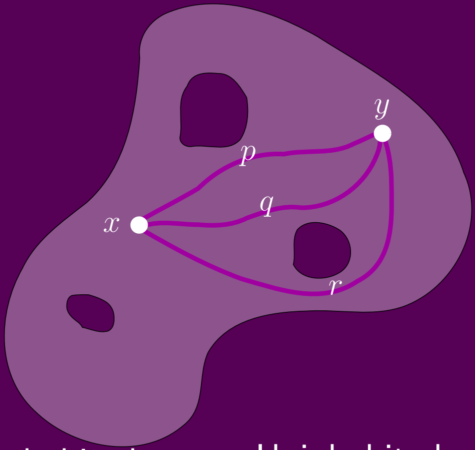
$$\mathsf{K}_X : \prod_{x:X}$$



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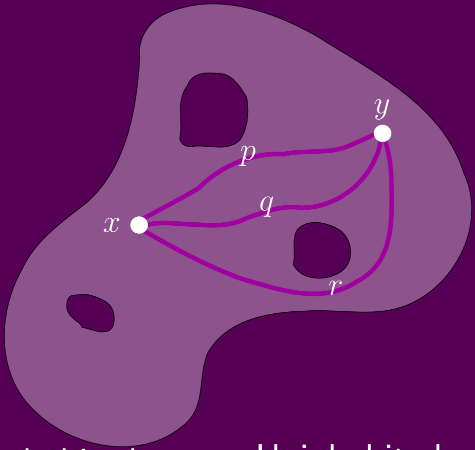

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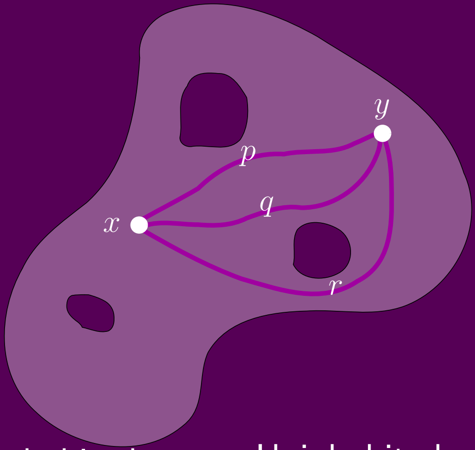
$$\mathsf{K}_X : \prod_{x:X} \prod_{p:x=x} p = \mathsf{refl}_x$$



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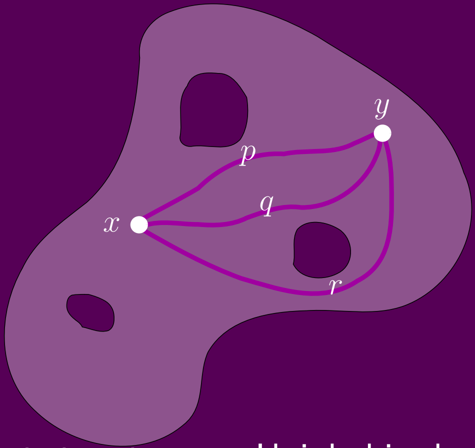
UIP_X :



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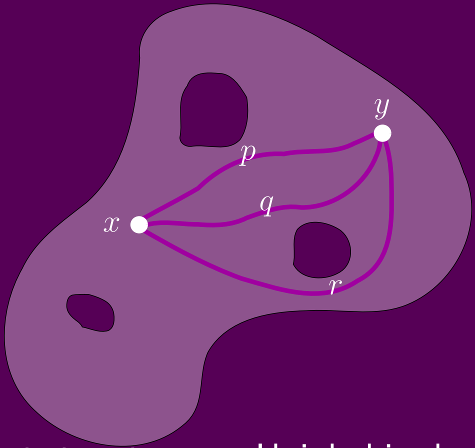

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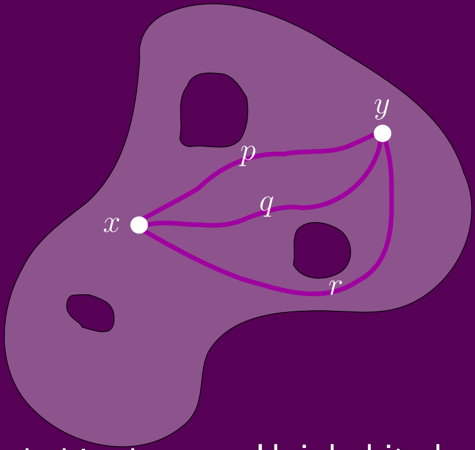

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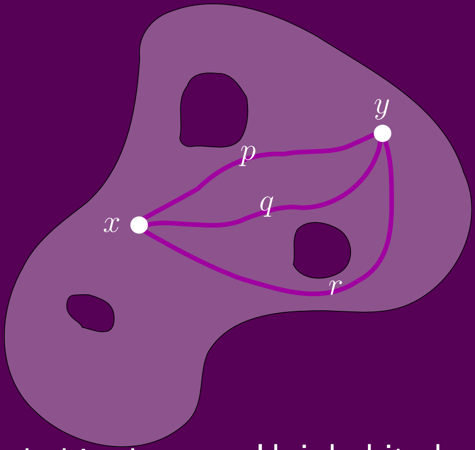

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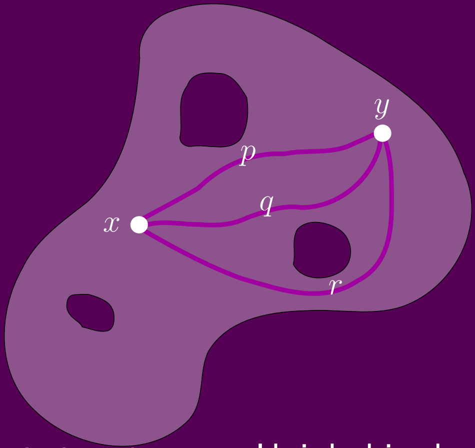

Thm.* If X : Type satisfies Axiom K, then there is a term of type

$$\prod_{x,y:X} \neg\neg(x = y) \to (x = y)$$

Inhabited:

$x = y$
$p = q$

Uninhabited:

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$q = r$

Starting-point of HoTT: Reject K

- In HoTT, we do not assume that K/UIP holds in general (though it does for many types, like $\mathbb{2}$)

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{-# OPTIONS --without-K #-}
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- In HoTT, types satisfying K/UIP are called "sets"

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```
{-# OPTIONS --without-K #-}
```

- In HoTT, types satisfying K/UIP are called “sets”
- Much of the research in HoTT is into “higher inductive types” (HITs), which are inductively-given types which have constructors for building non-refl identities

An example of a HIT: the circle

Classical definition:

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$$

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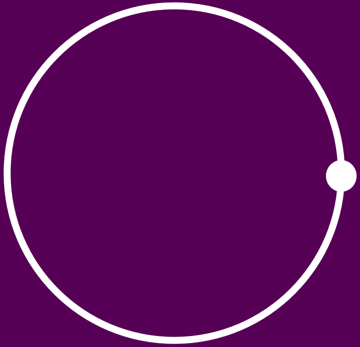

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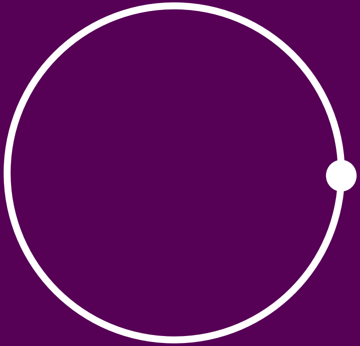
HoTT definition:



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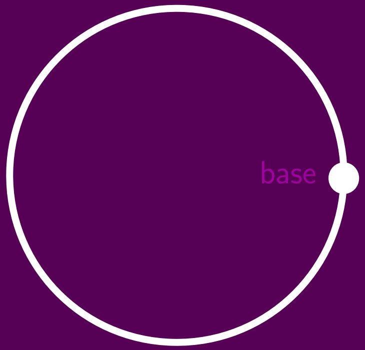
HoTT definition:

- base : $\mathbb{S}^1$



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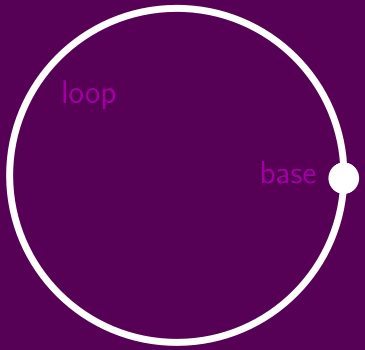
HoTT definition:

- base : $\mathbb{S}^1$
- loop : base $=$ base



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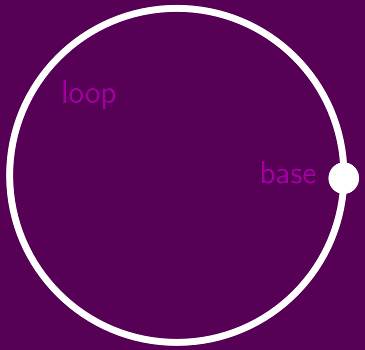
HoTT definition:

- $\mathsf{base} : \mathbb{S}^1$
- $\mathsf{loop} : \mathsf{base} = \mathsf{base}$

Inhabited: $\mathsf{base} = \mathsf{base}$, $\mathsf{loop} = \mathsf{loop}$, $\mathsf{loop} \cdot \mathsf{loop}^{-1} = \mathsf{refl}_{\mathsf{base}}, \ldots$



An example of a HIT: the circle

Classical definition:

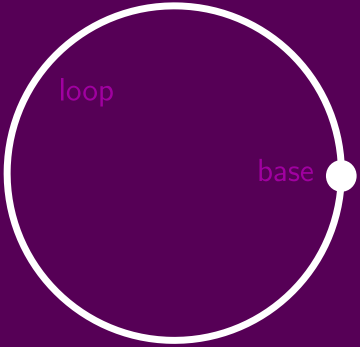
$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$$

HoTT definition:

- base : $\mathbb{S}^1$
- loop : base = base

Inhabited: base = base, loop = loop, loop $\cdot$ loop^{-1} = refl$_{\text{base}}$, ...

Uninhabited: loop = refl$_{\text{base}}$, loop = loop^{-1}, loop = loop $\cdot$ loop,...



Thank you!

Email me at jacobneu@andrew.cmu.edu

if you want to learn more HoTT!