Category Theory (80-413/713) F20 HW10, Exercise 4 Solution

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Problem:

> Let $\mathbb{C}$ be a small category and $\widehat{\mathbb{C}}$ the category of presheaves on $\mathbb{C}$. Let $F : I \to \widehat{\mathbb{C}}$ be a small diagram of presheaves.
>
> (a) Suppose that $\lim_i F_i$ exists and use the Yoneda Lemma to explicitly describe this presheaf.
>
> (b) Verify that this presheaf has the expected universal property of $\lim_i F_i$ and deduce that I-limits exist in $\widehat{\mathbb{C}}$.
>
> (c) Suppose that $\mathbb{C}$ has I-limits, prove that they are preserved by the Yoneda embedding $\mathbf{y} : \mathbb{C} \to \widehat{\mathbb{C}}$

Solution:

1

First, a brief bit about notation: I'll generally write the Yoneda embedding as $\mathbf{y}$ (e.g. writing $\mathbf{y}(X)$ for the presheaf $\mathsf{Hom}(-, X)$ instead of $\widehat{X}$), since I find it difficult to unambiguously use the notation $\widehat{X}$ when X is a long expression. [1] I'll try to give some indication of how to translate from this convention to the $\widehat{(-)}$ convention from lecture (also note that the book also uses $\mathbf{y}$ for the Yoneda embedding). For the category of presheaves on $\mathbb{C}$, I'll match the notation from lecture and write $\widehat{\mathbb{C}}$.

Next, a note about what we're doing. Usually, when working with limits in the category of presheaves $\widehat{\mathbb{C}}$, we'll generally *define* the presheaf $(\lim_i F_i) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ pointwise, i.e. put

Pointwise Defn

$$(\lim_i F_i)(X) \overset{\text{def}}{=} \lim_i (F_i(X)) \qquad (X \in \mathrm{Ob}(\mathbb{C}))$$

$$(\lim_i F_i)(u) \overset{\text{def}}{=} \lim_i (F_i(u)) \overset{\text{def}}{=} \langle F_i u \circ p_i^Y \;\mid\; i \in I \rangle \qquad (u : X \to Y \text{ in } \mathbb{C})$$

where $p_j^Z : \lim_i (F_i(Z)) \to F_j(Z)$ is the j-th projection function, a component of the limit cone on the diagram

$$F_-(Z) : I \to \mathsf{Set}$$

for each object Z of $\mathbb{C}$. In Part (a) of this problem, we are showing that (up to isomorphism) this is the *right* definition for $\lim_i F_i$. More precisely, the goal of (a) is to show that, up to isomorphism, **Pointwise Defn** is the *only* way to define $\lim_i F_i$ such that $\lim_i F_i$ is actually a limiting object (in $\widehat{\mathbb{C}}$) for the F_is.

To do this, we'll temporarily pretend that $\lim_i F_i$ is *not* defined by **Pointwise Defn**.

Instead, we take as given that $\lim_i F_i$ exists, i.e. we have a presheaf $\lim_i F_i$ and a cone in $\widehat{\mathbb{C}}$ consisting of natural transformations

$$\rho_j : \lim_i F_i \to F_j$$

that satisfies the universal mapping property of limits. Contrast that with the functor defined pointwise, which we'll now call L:

Defn L Define $L : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ by

$$L(X) \stackrel{\text{def}}{=} \lim_i (F_i(X)) \qquad (X \in \mathrm{Ob}(\mathbb{C}))$$

$$L(u) \stackrel{\text{def}}{=} \lim_i (F_i(u)) \stackrel{\text{def}}{=} \langle F_i u \circ p_i^Y \mid i \in I \rangle \qquad (u : X \to Y \text{ in } \mathbb{C})$$

We can think of (a) and (b) as a kind of "uniqueness" proof, where we're proving that there exists a unique way to define $\lim_i F_i$ that satisfies the UMP of the limit. (b) is the "existence" part of the proof: we show that there exists a presheaf satisfying the universal property, i.e. **Defn L** is *an* appropriate way of defining $\lim_i F_i$. (a) is the "uniqueness" part of the proof: if we have that $\lim_i F_i$ exists and satisfies the UMP of the limit of F, then $\lim_i F_i \cong L$, so, up to natural isomorphism, **Defn L** is the *only* appropriate way of defining $\lim_i F_i$. Thus **Pointwise Defn** is justified.

On the other hand, (c) is a related but slightly different statement. (a) and (b) give the formula for limits in $\widehat{\mathbb{C}}$, and, notice, do not make any assumptions about whether $\mathbb{C}$ itself has I-shaped limits. The existence of limits of shape I in $\widehat{\mathbb{C}}$ is unconditional: limits exist in $\widehat{\mathbb{C}}$ regardless of whether they exist in $\mathbb{C}$. **2** *But if $\mathbb{C}$ does have I-shaped limits*, then we'll argue that the Yoneda embedding preserves I-shaped limits, i.e. for any $G : I \to \mathbb{C}$, we have $\lim_i \mathbf{y}(G(i)) \cong \mathbf{y}(\lim_i G(i))$. **3** In a slogan:

limits of representables are represented by limits.

Note that in the equation $\lim_i \mathbf{y}(G(i)) \cong \mathbf{y}(\lim_i G(i))$, the limit on the right-hand side is being taken in $\mathbb{C}$ (hence the assumption that I-shaped limits exist in $\mathbb{C}$) and the limit on the left-hand side is being taken in $\widehat{\mathbb{C}}$ (so we'll use our work from (a) and (b)).

2

Enough talk, let's do some math. Suppose we have $F : I \to \widehat{\mathbb{C}}$ such that $\lim_i F_i$ exists. That is, we have a presheaf $(\lim_i F_i) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ and natural transforms $\rho_j : \lim_i F_i \to F_j$ satisfying the universal mapping property of limits in $\widehat{\mathbb{C}}$. Among the numerous other properties of limits we've proved, we have the following.

Limit-Homset Formula For any presheaf $Z \in \widehat{\mathbb{C}}$, there is a bijection

$$\mathsf{Hom}(Z, \lim_i F_i) \cong \lim_i \mathsf{Hom}(Z, F_i)$$

natural in Z.

Combining this with the Yoneda Lemma, we can articulate what the object part of $\lim_i F_i$ needs to be: for any object X of $\mathbb{C}$,

$$\begin{aligned}
(\lim_i F_i)(X) &\cong \mathsf{Hom}(\mathbf{y}(X), \lim_i F_i) && \text{(Yoneda Lemma)}\\
&\cong \lim_i \mathsf{Hom}(\mathbf{y}(X), F_i) && \text{Limit-Homset Formula}\\
&\cong \lim_i (F_i(X)) && \text{(Yoneda Lemma)}\\
&= L(X) && \text{Defn } L
\end{aligned}$$

So we get that the action of $\lim_i F_i$ on objects must essentially be that of L. 4

To see that the action of $\lim_i F_i$ on morphisms must be essentially the same as that of L, observe that each isomorphism in the chain above witnessing that $(\lim_i F_i)(X) \cong \lim_i(F_i(X))$ is natural in X: we know that the bijection from the Yoneda Lemma is natural in X, and the Limit-Homset Formula is natural in its argument. 5 So we have a natural iso

$$\Phi : \lim_i F_i \xrightarrow{\sim} L.$$

In particular, for any morphism $u : X \to Y$ in $\mathbb{C}$, we have:

$$\Phi_X \circ (\lim_i F_i)(u) = L(u) \circ \Phi_Y.$$

Or, remembering that Φ_X is an iso and that $L(u) = \lim_i(F_i(u))$, we get:

$$(\lim_i F_i)(u) = \Phi_X^{-1} \circ \lim_i (F_i(u)) \circ \Phi_Y.$$

Thus, **the morphism part of $\lim_i F_i$ is the same as L (up to natural iso).** 6 In conclusion: in order to satisfy the universal property of limits, $\lim_i F_i$ must essentially be defined pointwise.

3

(b) is more straightforward: we have the presheaf $L : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ defined pointwise using limits in Set, and we want to show it satisfies the universal property of limits. Start by defining the projections: for each $i \in I$, define $\rho_i : L \to F_i$ by $(\rho_i)_X = p_i^X$, the i-th projection out of $L(X) = \lim_i(F_i(X))$. This is indeed natural: the square

$$\begin{tikzcd}
L(X) \arrow[r, "(\rho_j)_X"] & F_j(X) \\
L(Y) \arrow[u, "L(u)"] \arrow[r, "(\rho_j)_Y"'] & F_j(Y) \arrow[u, "F_j(u)"']
\end{tikzcd}$$

commutes for each $u : X \to Y$ in $\mathbb{C}$ and each j because $L(u)$ is defined as $\langle F_i(u) \circ p_i^Y \mid$

$i \in I\rangle = \langle F_i(u) \circ (\rho_i)_Y \mid i \in I\rangle$, and

$$\begin{aligned}
&(\rho_j)_X \circ \langle F_i(u) \circ (\rho_i)_Y \mid i \in I\rangle \\
&= p_j^X \circ \langle F_i(u) \circ (\rho_i)_Y \mid i \in I\rangle \\
&= F_j(u) \circ (\rho_j)_Y
\end{aligned}$$

where the last step follows from the commutativity of the triangles in the limit cone on the diagram $F_{(-)}(X) : I \to \mathsf{Set}$. It's similarly easy to check that the ρ_j maps constitute a cone on F: for any $s : i \to j$ in I and any object X of $\mathbb{C}$,

$$\begin{aligned}
(F(s) \circ \rho_i)_X &= (Fs)_X \circ (\rho_i)_X && \text{(Composition of nat. transforms)} \\
&= (Fs)_X \circ p_i^X \\
&= p_j^X && \text{(the } p^X \text{ are a cone on } F_{(-)}(X)) \\
&= (\rho_j)_X
\end{aligned}$$

so, **since the X components of $F(s) \circ \rho_i$ and ρ_j are the same for every X, $F(s) \circ \rho_i = \rho_j$** **7** and thus the ρ_js constitute a cone on F.

To see that L and the ρ_js form a limit cone, pick another cone

$$\eta_j : P \to F_j.$$

It suffices to define what $\langle \eta_i \mid i \in I\rangle : P \to L$ is, and show that it satisfies the **necessary properties.** **8** We define it pointwise: for X an object of $\mathbb{C}$,

$$\langle \eta_i \mid i \in I\rangle_X = \langle (\eta_i)_X \mid i \in I\rangle : P(X) \to L(X).$$

First, we must show that $\rho_j \circ \langle \eta_i \mid i \in I\rangle = \eta_j$ for each j in I. Again, this is done componentwise:

$$\begin{aligned}
&(\rho_j)_X \circ \langle \eta_i \mid i \in I\rangle_X \\
&= p_j^X \circ \langle (\eta_i)_X \mid i \in I\rangle \\
&= (\eta_j)_X
\end{aligned}$$

as desired. Finally, it suffices to show that for $\psi : P \to L$, we have $\psi = \langle \rho_i \circ \psi \mid i \in I\rangle$. But this is the case if, for every X,

$$\psi_X = \langle (\rho_i \circ \psi)_X \mid i \in I\rangle.$$

But this equation holds by the **universal property of $L(X)$ as the limit of the $F_i(X)$s.** **9** So we're done.

Since I and F were arbitrary, we can conclude that $\widehat{\mathbb{C}}$ has all small limits.

4

So we've shown that Pointwise Defn is how limits in $\widehat{\mathbb{C}}$ *must* be defined, so from here on out we'll take that to be the definition of $\lim_i F_i$. Now let's turn our attention to (c): suppose we instead have an I-shaped diagram G in $\mathbb{C}$ itself such that $\lim_i G(i)$ exists.

We want to show that the Yoneda embedding preserves this limit, i.e. that

$$\lim_i \mathbf{y}(G(i)) \cong \mathbf{y}(\lim_i G(i)).$$ 3

So it doesn't matter if we take the limit in $\mathbb{C}$ and then use Yoneda to embed it into $\widehat{\mathbb{C}}$, or embed it in $\widehat{\mathbb{C}}$ first and then take the limit – either way we get the same result.

To do this, again take an arbitrary object X of $\mathbb{C}$ and observe:

$$\begin{aligned}
&(\lim_i \mathbf{y}(G(i)))(X) \\
&= \lim_i(\mathbf{y}(Gi)(X)) && \text{Pointwise Defn} \\
&= \lim_i \mathsf{Hom}(X, Gi) \\
&\cong \mathsf{Hom}(X, \lim_i Gi) && \text{Limit-Homset Formula} \\
&\cong \mathsf{Hom}(\mathbf{y}(X), \mathbf{y}(\lim_i Gi)) && (\text{Yoneda } 10) \\
&\cong (\mathbf{y}(\lim_i Gi))(X). && (\text{Yoneda } 10)
\end{aligned}$$

We can check that this is natural in X, again by the naturality of the Limit-Homset Formula and the naturality of the Yoneda Lemma. So we have exhibited a natural iso between $\lim_i \mathbf{y}(Gi)$ and $\mathbf{y}(\lim_i Gi)$, so, since G and I were arbitrary, $\mathbf{y}$ preserves all small limits.

Notes:

1 For instance, the LaTeX code

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(abcd(\widehat{efg(hij(klm)no)pq}rs)tu)vwxyz
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produces this:

$$(abcd(efg(hi\widehat{j(klm})no)pqrs)tu)vwxyz$$

Note that the hat is nowhere near the 'e', where it's nominally supposed to begin. There are some workarounds (e.g https://tex.stackexchange.com/questions/100574/really-wide-hat-symbol), but this is why I like $\mathbf{y}$. A similar issue arises for the over-tildes used for the exponential transpose ("currying"), hence why I also changed that to usual functor notation in my lambda calculus lecture.

2 This is one of the major reason we're interested in $\widehat{\mathbb{C}}$: regardless of whether $\mathbb{C}$ has limits, colimits, and exponentials, $\widehat{\mathbb{C}}$ has all those things. So, in order to prove stuff about $\mathbb{C}$, we can instead use all the nice structure of $\widehat{\mathbb{C}}$ to prove the result we want about objects of $\widehat{C}$, and then use the Yoneda Principle to transfer these consequences back to $\mathbb{C}$.

This is very much analogous to the classic mathematical technique of (a) taking a problem which cannot be easily solved in the real numbers and viewing it in the complex numbers, (b) using the extra structure of the complex numbers to find a *real* solution to the problem, and then (c) forgetting about the intervening complex-numbers-y stuff and just taking the real solution as-is. This is useful for pure math, and also for applied stuff: for example, lots of electrical engineering involves solving problems in complex numbers (because $e^{i\theta}$ has much nicer algebraic properties than $\sin(\theta)$ and $\cos(\theta)$), and then taking the real part at the end of the day, as if you were working with real cosines the whole time. In category theory, we take problems which are hard to solve natively in $\mathbb{C}$, inject them (using the Yoneda embedding) into $\widehat{\mathbb{C}}$ to solve them using the extra features of $\widehat{\mathbb{C}}$, and then argue that the solution we get is "real", i.e. representable, and therefore existed in $\mathbb{C}$ all along.

3

$$\lim_i \widehat{G(i)} \cong \widehat{\lim_i G(i)}$$

4 For (a), this is basically all I was looking for, because it articulates in explicit detail what $(\lim_i F_i)(X)$ must be (up to iso) for each object X. In particular, it says essentially what the object part of $\lim_i F_i$ must be, and specifies in terms of things we already know how to do (applying the functors F_i and taking limits in Set).

Several of you stopped just short of this conclusion, e.g. concluding that

$$(\lim_i F_i)(X) \cong \lim_i \mathsf{Hom}(\mathbf{y}(X), F_i).$$

While this does show some of the key steps and does give me more information about what $\lim_i F_i$ is, this is not a finished answer. $\lim_i \mathsf{Hom}(\mathbf{y}(X), F_i)$ is still a very sophisticated expression (a limit of sets of natural transformations of presheaves), and difficult to calculate explicitly (figuring out what $\mathsf{Hom}(\mathbf{y}(X), F_i)$ is directly without first converting it to $\lim_i(F_i(X))$ can be quite hard in general).

5 The composition of natural transforms is again a natural transform (paste together the naturality squares).

6 As with the objects, specifying the morphism part (up to conjugation with the natural transformation Φ) is enough to tell us everything we need to know about $\lim_i F_i$.

7 I'm using a kind of "extensionality" for natural transforms: if β and η are natural transforms such that

$$\beta_X = \eta_X$$

for all objects X of the appropriate category ($\mathrm{dom}(\mathrm{dom}(\beta))$), then

$$\beta = \eta.$$

In other words, all there is to a natural transformation are its components.

8 Another way to phrase the universal property of limits: given a J-shaped diagram E and a cone $(p_j : L \to E(j))_{j \in J} \in \mathsf{Cone}(E, L)$, the following are equivalent:

1. $(p_j)_{j \in J}$ is a limit cone for E
2. There exists an operation

$$\langle - \rangle : \mathsf{Cone}(E, Z) \to \mathsf{Hom}(Z, L)$$

such that

$$p_{j_0} \circ \langle f_j \mid j \in J \rangle = f_{j_0} \qquad \text{(For all cones } (f_j : Z \to E(j))_{j \in J} \text{ and all } j_0 \in J)$$

$$\langle p_j \circ k \mid j \in J \rangle = k \qquad \text{(For all } k : Z \to L)$$

9 Formulated as in **8**.

10 The first of these uses the 'stronger' form of the Yoneda Lemma (that $\mathsf{Hom}(\mathbf{y}(-), F) \cong F$ for all presheaves F), while the second uses the 'weaker form' (that $\mathbf{y}$ is full and faithful). I generally like to call the latter as the 'Yoneda Theorem', which is proved using the stronger form, which to me is *the* Yoneda Lemma.