This document lists common errors students made in Homework 1 – particularly in Problems 3 & 4 –, and some tips to avoid them. Though many of these are about induction specifically, hopefully some of these tips will make you better at writing proofs in general. Here's another list of good tips about writing nice proofs.

One thing to note: several of these suggestions are arguably more about style (e.g. specifying how we'd like to see proofs structured, what elements to include in certain proofs, certain ways of phrasing things), than content. So your proof is not necessarily wrong if it doesn't follow these guidelines. For instance, most professional mathematicians will rarely state their inductive hypotheses explicitly in a proof (and you may not have in other classes), and that's fine if you know what you're doing. But if you're less experienced writing (structural) induction proofs, it might be a good idea to explicitly state your inductive hypothesis: we'll deduct points from your proof if it's unclear what your IH is, so might as well be safe. So I'd take these tips as guidelines, unless you're comfortable enough to not need them.

Also: special thanks to people who typeset their homework submissions (especially in LaTeX)! If you ever need help typesetting your homeworks (or don't know how but would like to learn), stop by my office hours or send me an email. Thanks! -Jacob

1 Listing Steps In The Logical Order

If you're writing a mathematical proof (and I'm reading it), your task is to walk me through a series of logical steps and, by the end, convince me of the truth of a mathematical statement. So your proof should begin with statements I'd believe (definitions, obvious truths like $1 = 1$, previously proven results, explicit assumptions) and end with the statement you want to prove. So, if you're proving $(x+1)^2 \geq (x-1)^2$ for $x \geq 0$, I'd much rather see:

$$\begin{aligned}(x+1)^2 &= x^2 + 2x + 1 \\ &\geq x^2 - 2x + 1 && (x \geq 0) \\ &= (x-1)^2\end{aligned}$$

than

$$\begin{aligned}(x+1)^2 &\geq (x-1)^2 \\ x^2 + 2x + 1 &\geq x^2 - 2x + 1 \\ 2x &\geq -2x \\ 4x &\geq 0 \\ x &\geq 0\end{aligned}$$

They contain the same information, but the first one conveys it in the logically-precise order: a series of steps which I'd individually believe and which together prove the desired statement. The second one, read literally, starts by assuming with what we're trying to prove $(x+1)^2 \geq (x-1)^2$ and then derives a proof of what we're assuming (that $x \geq 0$). Whenever you need to do a series of equations/inequalities, do the first one!

2 Explicitly Noting Your Steps

In order for your proof to convince me, your reader, of the truth of the statement, I need to understand what's going on with every stage of your proof. Make it easy for me: tell me what you're doing! If you look at a complicated proof in a math textbook, it'll be something like:

> *Proof.*
> I want to prove P. I'll do it by first proving Q, and then proving that Q implies P.
>
> First, let's prove Q. I'll use method X...
> . [Proof of Q using method X]
> So we've proved Q.
>
> Now let's prove Q implies P. Since Q is true ...
> . [Proof of P assuming Q]
> ...therefore P.
>
> $\square$

Mimic this: say what you're going to prove, why proving *that* suffices to prove what you're asked to prove, and how you're going to prove it.

A good convention which many people adopt in their proofs is writing "**Want to show:**" (commonly abbreviated WTS) and then saying what they're trying to prove before they prove it. This gives me, the person grading your proof, a roadmap of what you're doing. You'll notice in the proof templates below that I do this.

One other note: **be explicit about what you're assuming**. If you use any fact without directly proving it, you should explicitly introduce that assumption, and then note every time you use it. A common example is the inductive hypothesis in an induction proof: it's a really good idea to start the inductive step by saying explicitly what your IH is, and then putting a little (IH) citation everywhere you use it. Otherwise it seems like you're just pulling this assumption out of thin air.

3 Labelling and Justifying Steps

Notice in the above proof, I leave a comment ($x \geq 0$) off to the side of one of the steps. I did this because the step it's next to (that $x^2 + 2x + 1 \geq x^2 - 2x + 1$) is not always true: I need to note that $x \geq 0$ to justify that step. I highly recommend doing this: labelling and separately justifying steps is an elegant way to present a proof with several non-trivial steps. Here's a more sophisticated example:

Want to show: $[\varphi] = [(\neg\varphi) \to \bot]$. Pick an arbitrary valuation v.

$$\begin{aligned} [\![\varphi]\!]_v = \mathit{true} &\iff [\![\neg\varphi]\!]_v = \mathit{false} && \text{(defn of } [\![-]\!] \text{ for } \neg) \\ &\iff [\![(\neg\varphi) \to \bot]\!]_v = \mathit{true} && (\star) \end{aligned}$$

Where step $(\star)$ is true because $[\![(\neg\varphi) \to \bot]\!]_v$ is true iff $[\![\neg\varphi]\!]_v = \mathit{false}$ or $[\![\bot]\!]_v = \mathit{true}$ (by defn of $[\![-]\!]$ for $\to$), and $[\![\bot]\!]_v$ has to be false.
Since v was arbitrary, this proves $\models \varphi \leftrightarrow ((\neg\varphi) \to \bot)$, which establishes the claim.

Notice how this allowed me to maintain the nice chain of $\iff$'s central to the proof, and still explain the more subtle reasoning behind the last $\iff$ in the chain.

4 Outlining the Structure of the Proof

If you're using a certain proof technique, tell me that's what you're doing and label the components of that kind of proof. Here's templates for a few common styles of proof.

4.1 Double Containment

I'll show $X = Y$ by double containment.

- Pick arbitrary $x \in X$ ... therefore $x \in Y$
- Pick arbitrary $y \in Y$ ... therefore $y \in X$

Unless it's really short, I'd format these as two separate paragraphs.

4.2 Well-Definedness

Suppose $\sim$ is an equivalence relation on X, and I have a definition a function $f : X/\sim \to Y$ which I want to prove is well-defined.

Let x, x' be arbitrary elements of X, and suppose $x \sim x'$. I want to show that this implies

$$f([x]) = f([x'])$$

where $[x]$ is, as usual, the $\sim$-equivalence class of x.

(Proof that $f([x]) = f([x'])$ using the given definition of f, and the assumption that $x \sim x'$)

Therefore, since x, x' were arbitrary, $x \sim x'$ implies $f([x]) = f([x'])$ for all $x, x' \in X$. Thus f is well-defined.

4.3 Weak Induction

I prove $P(n)$ for all $n \in \mathbb{N}$, by induction on n.
BC: $n = 0$
(Proof of $P(0)$)
IS: $n = k + 1$ for *some* $k \in \mathbb{N}$
IH: Suppose $P(k)$
Want To Show: $P(k + 1)$
(Proof of $P(k + 1)$, assuming $P(k)$)

Notice that I note *what* I'm inducting on (n) and what style of induction (weak).

4.4 Strong Induction

I prove $P(n)$ for all $n \in \mathbb{N}$, by strong induction on n.
BC: $n = 0$
(Proof of $P(0)$)
IS: $n = k + 1$ for *some* $k \in \mathbb{N}$
IH: Suppose $P(0), P(1), \ldots, P(k)$
Want To Show: $P(k + 1)$
(Proof of $P(k + 1)$, assuming $P(0), P(1), \ldots, P(k)$)

Notice that I note *what* I'm inducting on (n) and what style of induction (strong).

4.5 Structural Induction

I prove $P(\varphi)$ for all propositional formulas φ, by structural induction on φ.
BC: $\varphi = p$ for p atomic
(Proof of $P(p)$)
IH: Suppose $P(\varphi)$ and $P(\psi)$ for some abritrary formulas φ, ψ

IS $\neg$: Want to show: $P(\neg\varphi)$
(Proof of $P(\neg\varphi)$, assuming $P(\varphi)$)
IS $\wedge$: Want to show: $P(\varphi \wedge \psi)$
(Proof of $P(\varphi \wedge \psi)$, assuming $P(\varphi)$ and $P(\psi)$)
IS $\vee$: Want to show: $P(\varphi \vee \psi)$
(Proof of $P(\varphi \vee \psi)$, assuming $P(\varphi)$ and $P(\psi)$)
IS $\rightarrow$: Want to show: $P(\varphi \rightarrow \psi)$
(Proof of $P(\varphi \rightarrow \psi)$, assuming $P(\varphi)$ and $P(\psi)$)

Often we can reduce redundancy, e.g. by proving $P(\varphi * \psi)$ for a generic binary connective $*$.

4.6 Contradiction

> I'm trying to prove that P is false. Assume for the sake of contradiction that P. Then ... (Proof of an absurd statement, assuming P) ... But this is a contradiction! So we conclude that P is false.

or

> I'm trying to prove that P is true. Assume for the sake of contradiction that P is false. Then ... (Proof of an absurd statement, assuming not P) ... But this is a contradiction! So we conclude that P is true.

5 Types of Induction

One proof technique we've introduced in this class is *proof by structural induction*. This is just a special flavour of induction, specifically designed to prove statements about all propositional formulas. This is opposed to *weak* (or simple) induction and *strong* induction, which are techniques to prove statements about all natural numbers.

A few students were unclear about the distinction between these kinds of induction (or tried to combine them), so I wanted to clarify the difference.

5.1 Weak & Strong Induction

Weak and strong induction (which you probably learned in other classes) are tools to prove statements about *all natural numbers*. The formats for these proofs are given in 4.3 and 4.4. If you have questions about these, definitely come ask!

In most cases, these are not the proper techniques to prove statements about all propositional formulas. The set of all propositional formulas is defined inductively, and if you want to use that inductive construction to prove statements about all formulas **structural induction is the tool you want** (see below).

If you're interested, you *can* prove statements about all propositional formulas using strong induction, but you're not inducting on the formula itself (because formulas aren't natural numbers, and strong induction only works for natural numbers). Instead, you can do strong induction on the length of the formula, on the rank of the formula, etc. These quantities *are* natural numbers and every formula has a well-defined length & rank, so these proofs are fully rigorous (if I prove that $P(\varphi)$ holds for every φ of rank n, and prove that for every n, then I've proved it for every φ). However, at this level it's comparatively rare to encounter a statement which you can prove for all formulas φ by strong induction on rank(φ) which *cannot* be proven by structural induction on φ itself. And usually strong induction on length(φ) or rank(φ) is the exact same proof as the corresponding structural induction proof, but with added notational annoyance and extra steps. So use structural induction when you can (and, if you're interested, come ask and we can show you examples of proofs which can't be done structurally – they're kinda weird).

5.2 Structural Induction

The goal of structural induction is to prove statements **about all propositional formulas**. Propositional formulas are defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi$$

So the cases in your structural induction proof should mirror this exactly: every formula φ is either an atomic proposition, the negation of a propositional formula, or the combination of two formulas using one of the binary connectives. So, if you prove all the steps listed in 4.5, then you've proven the statement for all formulas.

Important to note: we assume $P(\varphi)$ and $P(\psi)$ as the inductive hypothesis! That's because, when proving the binary connective steps, we'll (usually) need both $P(\varphi)$ and $P(\psi)$ to be true for our induction to work. Don't forget to include both in your IH.