Lecture 6
Principles of Functional Programming
Summer 2020

Parallelism and Trees

Section 1

Recap: Work Analysis of Recursive Functions

The Tree Method

The Tree Method

0 How you're quantifying input size

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1 Recurrence

The Tree Method

0 How you're quantifying input size
1 Recurrence
2 Description of work tree

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0 How you're quantifying input size

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3 Measurements of work tree (height, and width at each level)

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4 Summation

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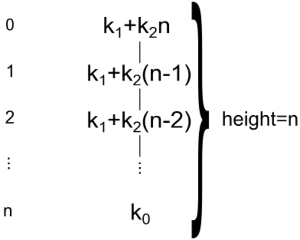
5 Big-O

1. Recurrence:

$$W(0) = k_0$$
$$W(n) = k_1 + k_2 n + W(n-1)$$

2. Work Tree



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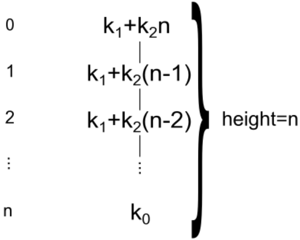

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$$W(n) \text{ is } O(n^2)$$

Section 2

Asymptotic Analysis of Multi-Step Algorithms

Sorting

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Or, recursively: a value `v:int list` is **sorted** if either `v=[]` or `v=[x]` for some `x`, or `v=x::x'::xs` where `Int.compare(x,x')` $\ncong$ `GREATER` and `x'::xs` is sorted.

Spec

6.0

```
fun isSorted ([]:int list):bool = true
  | isSorted [x] = true
  | isSorted (x::x’::xs) =
          (x<=x’) andalso isSorted(x’::xs)
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`sort : int list -> int list`
REQUIRES: true
ENSURES: `sort(L)` evaluates to a sorted permutation of `L`

A "permutation" of `L` is just a list that contains the same elements the same number of times as `L`, just in a possibly different order. So `[1,1,2,3]` is a permutation of `[3,1,2,1]` but not of `[3,2,1]`.

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We'll be focusing on *merge sort*, which consists of the following three steps:

1. Split the input list in half
2. Sort each half
3. *merge* the sorted halves together to obtain a sorted whole

Specs

`split : int list -> int list * int list`

REQUIRES: true

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```
fun split ([]:int list) = ([],[])
  | split [x] = ([x],[])
  | split (x::x'::xs) =
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       val (A,B) = split xs
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      (x::A,x'::B)
    end
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fun merge (L1:int list,[]:int list) = L1
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        merge(msort A,msort B)
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Analysis

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$$W_{\texttt{msort}}(n) \leq k_2 + k_3 n + W_{\texttt{msort}}\left(\frac{n}{2}\right) + W_{\texttt{msort}}\left(n - \frac{n}{2}\right) + k_4 n$$
$$\approx 2W_{\texttt{msort}}(n/2) + kn$$

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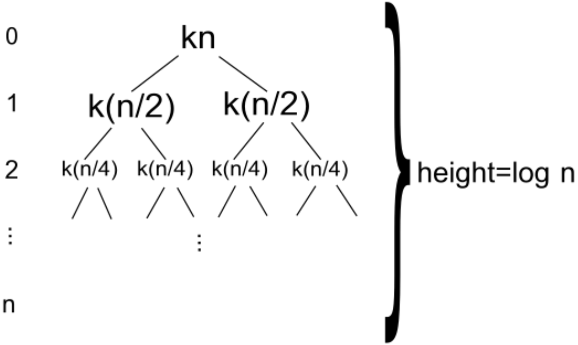

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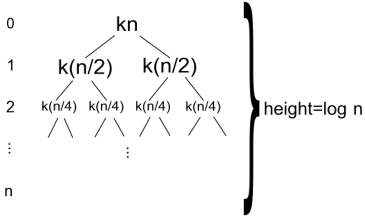

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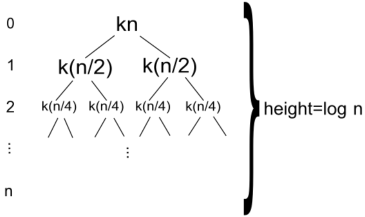

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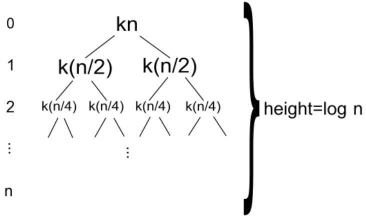

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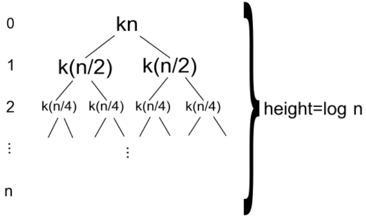

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5. Big O:

$$W(n) \text{ is } O(n \log n)$$

(pause for questions)

Section 3

Parallel Cost Analysis

Opportunity for parallelism

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- This is known as an "**opportunity for parallelism**"

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NOT an opportunity

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- We will express both as a big-O complexity class, representing how the runtime grows as the input size grows
- We will obtain both by analyzing the code, obtaining recurrences, and solving those recurrences (using the tree method) to obtain the big-O complexity

Calculating Work and Span

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Calculating Work and Span

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$$W_{\texttt{x}} = W_{\texttt{e1}} + W_{\texttt{e2}}$$

$$S_{\texttt{x}} = \max\left(S_{\texttt{e1}}, S_{\texttt{e2}}\right)$$

If we assume that `e1` and `e2` take approximately the same amount of time to evaluate, then

$$W_{\texttt{x}} = 2W_{\texttt{e1}} \qquad S_{\texttt{x}} = S_{\texttt{e1}} = S_{\texttt{e2}}$$

split doesn't have any parallelism

6.1

```
fun split ([]:int list) = ([],[])
  | split [x] = ([x],[])
  | split (x::x'::xs) =
    let
       val (A,B) = split xs
    in
      (x::A,x'::B)
    end
```

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\begin{aligned}
S_{\texttt{split}}(0) &= k_0 \\
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2 Work Tree...

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Height: $\log n$ Span on the i-th level: $\frac{kn}{2^i}$

4&5 Sum:

$$S(n) \approx \sum_{i=0}^{\log n} \frac{kn}{2^i} \le \sum_{i=0}^{\infty} \frac{kn}{2^i} = 2kn = O(n)$$

Conclusions

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- Making recursive calls to `msort` in parallel decreased runtime to $O(n)$ – the span
- Unable to take further advantage of parallelism, because `split` and `merge` only made one recursive call
- This is a shortcoming of `list`s themselves: they're an inherently sequential data structure and are thus limited in how much parallelism can be utilized

(pause for questions)

Section 4

Trees

Binary trees in SML

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- This declares a new type called `tree` whose constructors are `Empty` and `Node`. `Empty` is a *constant constructor* because it's just a value of type `tree`. `Node` takes in an argument of type `tree*int*tree` and produces another `tree`.
- All trees are either of the form `Empty` or `Node(L,x,R)` for some `x : int` (referred to as the *root* of the tree), some `L : tree` (referred to as the *left subtree*), and some `R : tree` (referred to as the *right subtree*)

Basic Quantities

Height (or *depth*):

6.6

```
fun height (Empty:tree):int = 0
  | height (Node(L,_,R)) =
      1 + Int.max(height L,height R)
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```

Size

6.7

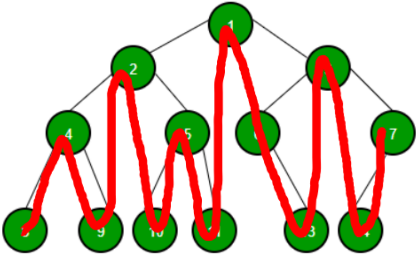
```
fun size (Empty:tree):int = 0
  | size (Node(L,_,R)) =
      1 + size L + size R
```

Traversals

Inorder

6.9

```
fun inord (Empty:tree):int list = []
  | inord (Node(L,x,R)) =
     (inord L) @ (x::inord R)
```

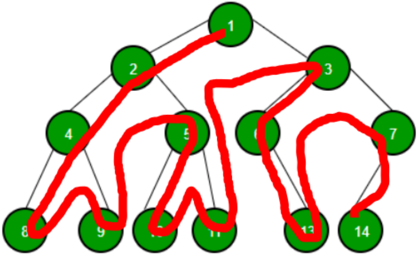


Traversals

Preorder

6.10

```
fun preord (Empty:tree):int list = []
  | preord (Node(L,x,R)) =
     x::((preord L) @ (preord R))
```



Minimum

6.11

```
fun min (Empty:tree, default:int) = default
  | min (Node(L,x,R),default) =
     let
       (* Parallel *)
       val (minL,minR) =
          (min(L,default), min(R,default))
     in
       (* Constant-time *)
       Int.min(x,Int.min(minL,minR))
     end
```

Depth-Analysis of `min`

0 Notion of size: depth d of the input tree

1 Recurrences:

$$\begin{aligned} W_{\texttt{min}}(0) &= k_0 \\ W_{\texttt{min}}(d) &\leq k_1 + 2W_{\texttt{min}}(d-1) \end{aligned}$$

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2-4 …

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If the input tree is **balanced**, then $2^d \approx n$, where n is the size (number of nodes)

Size-Analysis of `preord`

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Upcoming lectures

- Thursday: Structural induction on trees, datatypes
- Friday: Sorting with trees

Section 5

Lecture 6.5 : Trees in SML

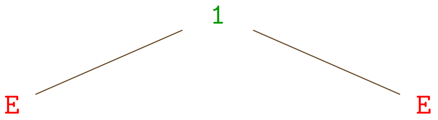
Section 6

Arboretum

`E`

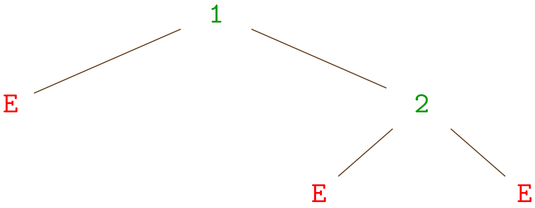
6.12

```
Empty
```



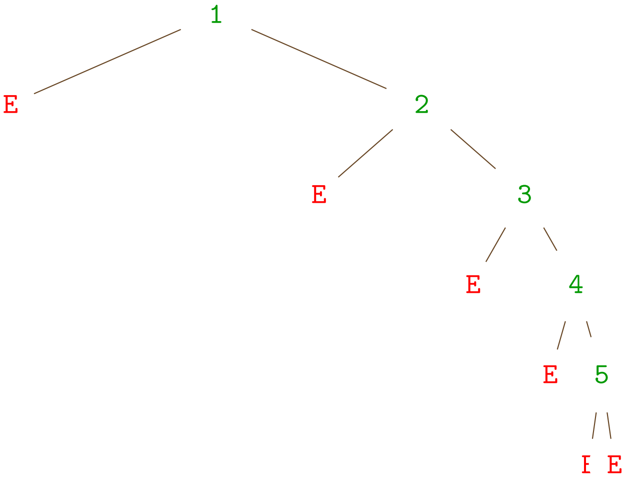
6.13

```
Node(Empty,1,Empty)
```



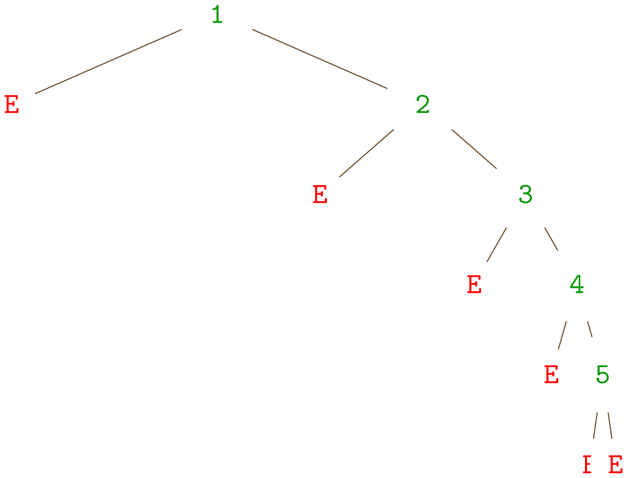
6.14

```
Node(Empty,1,Node(Empty,2,Empty))
```



6.15

```
Node(Empty,1,Node(Empty,2,Node(Empty,3,Node(
    Empty,4,Node(Empty,5,Empty)))))
```



6.15

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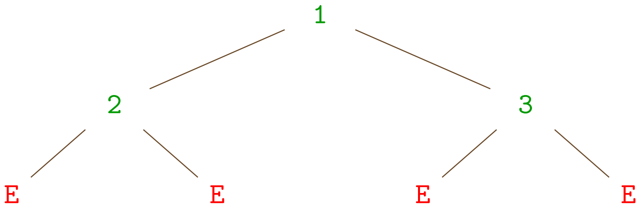
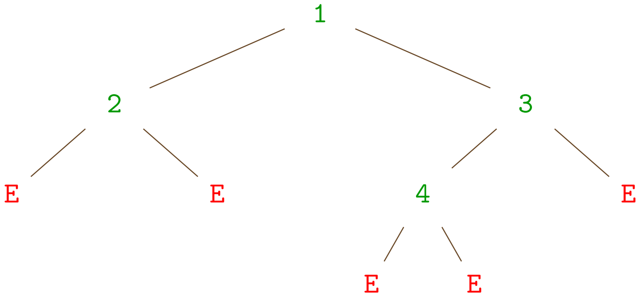


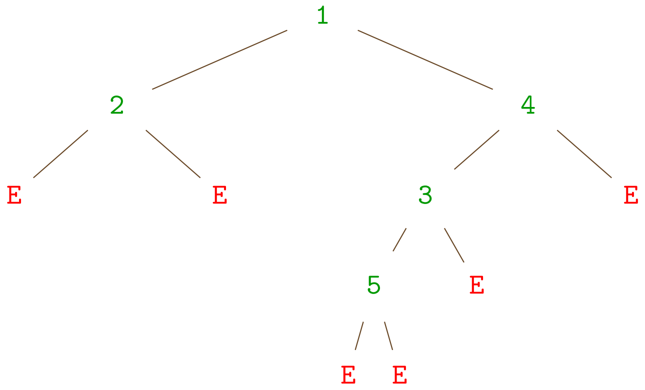
6.16

```
Node(Node(Empty,2,Empty),1,Node(Empty,3,Empty))
```



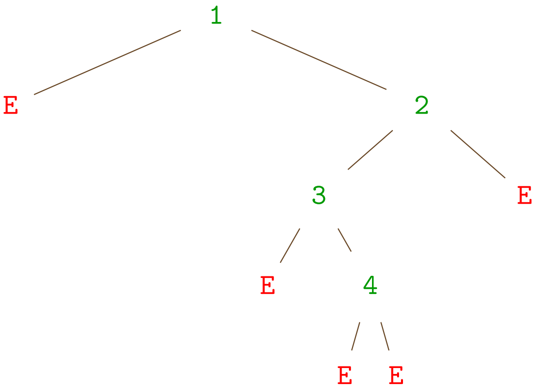
6.17

```
Node(Node(Empty,2,Empty),1,Node(Node(Empty,4,Empty),3,Empty))
```



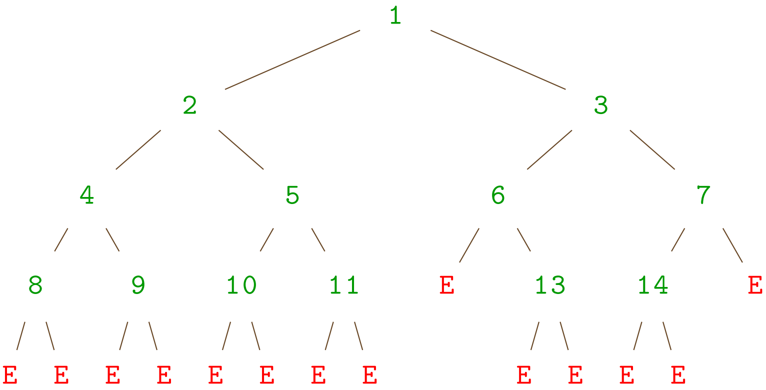
6.18

```
Node(Node(Empty,2,Empty),1,Node(Node(Node(Empty,5,Empty),3,Empty),4,Empty))
```



6.19

```
Node(Empty,1,Node(Node(Empty,3,Node(Empty,4,
    Empty)),2,Empty))
```



6.20

```
Node(Node(Node(Node(Empty,8,Empty),4,Node(
    Empty,9,Empty)),2,Node(Node(Empty,10,Empty)
    ,5,Node(Empty,11,Empty))),1,Node(Node(Empty
    ,6,Node(Empty,13,Empty)),3,Node(Node(Empty
    ,14,Empty),7,Empty)))
```

Thank you!