Intro to
Homotopy Type Theory

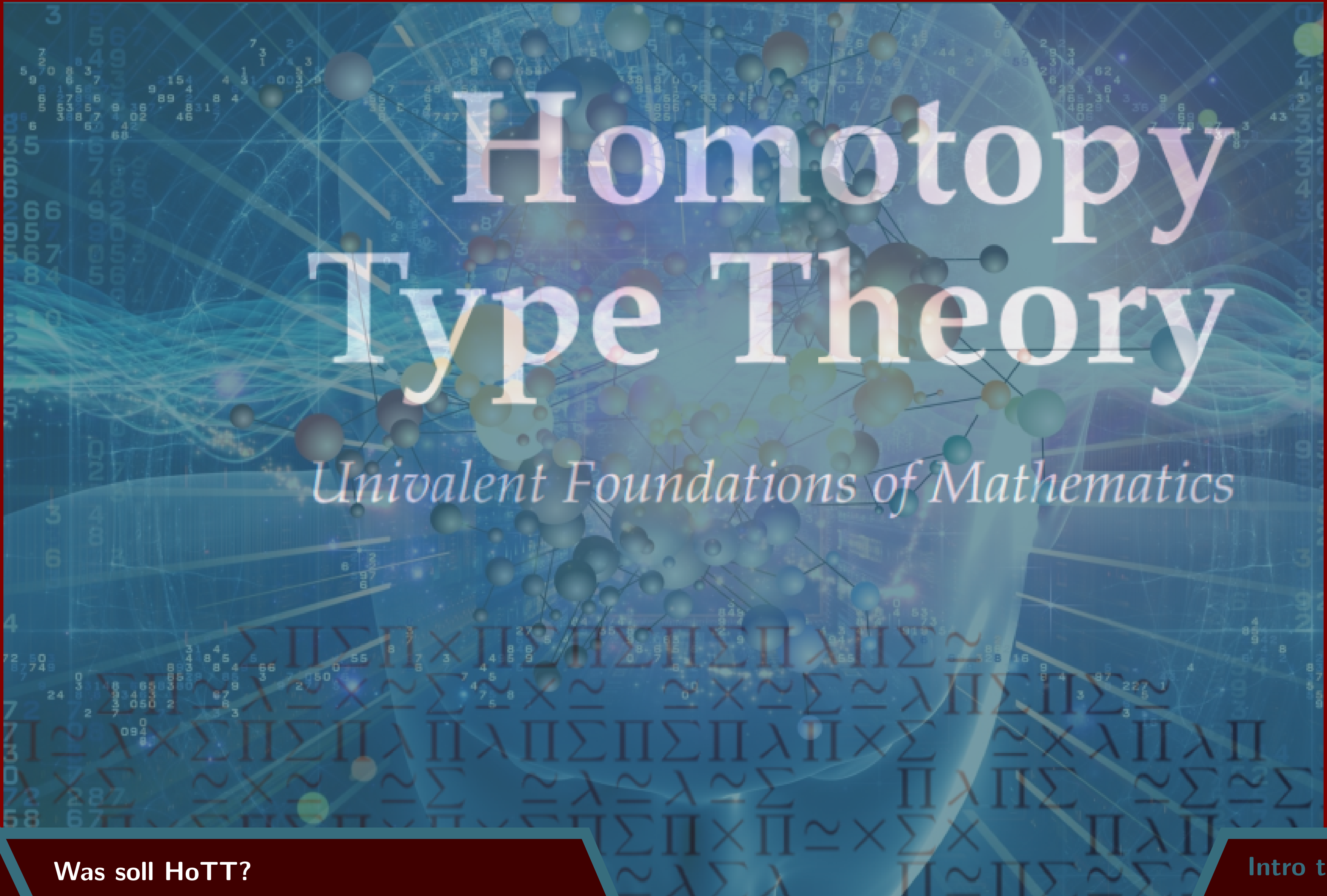
Was soll HoTT?

Was soll HoTT?

Was sind und was sollen die Zahlen (1888)

- *die Zahlen* = the (natural) numbers
- *was sind* = what are

- What?
- *Why?*


Homotopy
Type Theory
Univalent Foundations of Mathematics

0 A Problem with Proof-Reading

$\Delta S > 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

$$\mathrm{Hom}_R\left(\bigoplus_{i \in I} M_i, L\right) \cong \prod_{i \in I} \mathrm{Hom}_R(M_i, L).$$

$$\iint \ldots = \int \left(\int f(x,y)\,\mathrm{d}y\right) \mathrm{d}x = \int_Y \left(\int_X f(x,y \ldots\right) \ldots$$


Proof of famous theorem

Titans of Mathematics Clash Over Epic Proof of ABC Conjecture

Two mathematicians have found what they say is a hole at the heart of a proof that has convulsed the mathematics community for nearly six years.

Despite multiple conferences dedicated to explicating Mochizuki's proof, number theorists have struggled to come to grips with its underlying ideas. His series of papers, which total more than 500 pages, are written in an impenetrable style, and refer back to a further 500 pages or so of previous work by Mochizuki, creating what one mathematician, Brian Conrad of Stanford University, has called "a sense of infinite regress."

But the meeting led to an oddly unsatisfying conclusion: Mochizuki couldn't convince Scholze and Stix that his argument was sound, but they couldn't convince him that it was unsound. Mochizuki has now posted Scholze's and Stix's report on his website, along with several reports of his own in rebuttal. (Mochizuki and Hoshi did not respond to requests for comments for this article.)

What we need from our proofreaders

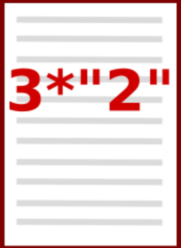
- Ability to read & understand complex mathematical arguments
- Endless patience
- Superhumanly-infallible meticulousness
- Always come to a decision
- Will work for free/cheap

What can be done?

1 The Art of Typechecking

Python-related meltdown

Goal: protect the user from code with type errors



Typechecking

So what?

2 Remaking Math in Type Theory's Image

Example Typing Rules

- If `e1` is a term of type `number` and `e2` is a term of type `number`, then

  ```
  e1 < e2
  ```

  is a term of type `boolean`
- If `e1` and `e2` are terms of type `string`, and `b` is a term of type `boolean`, then

  ```
  if b then e1 else e2
  ```

  is a term whose type is `string`

Crazy Idea

- Theorems are types
- Proofs are terms
- If the proof typechecks, it's correct

Example: Gauss's Kindergarten Formula, proved in the Lean Proof Assistant

Computer proof assistants

- ✓ Ability to read & understand complex mathematical arguments
- ✓ Endless patience
- ✓ Superhumanly-infallible meticulousness
- ✓ Always come to a decision
- ✓ Will work for free/cheap

Formalized Mathematics (example)

Univalent categories and the Rezk completion

Benedikt Ahrens, Chris Kapulkin, Michael Shulman

```
Lemma pre_comp_rezk_eta_is_ess_surj :
   essentially_surjective (pre_composition_functor A (Rezk_completion A) C (Rezk_eta A)).
Proof.
  apply pre_composition_essentially_surjective.
  assumption.
  apply Rezk_eta_essentially_surjective.
  apply Rezk_eta_is_fully_faithful.
Qed.

Theorem Rezk_eta_Universal_Property :
  isweq (pre_composition_functor A (Rezk_completion A) C (Rezk_eta A)).
Proof.
  apply equiv_of_cats_is_weq_of_objects.
  apply is_category_functor_category;
  assumption.
  apply is_category_functor_category;
  assumption.

  apply rad_equivalence_of_precats.
  apply is_category_functor_category;
  assumption.
  apply pre_comp_rezk_eta_is_fully_faithful.
  apply pre_comp_rezk_eta_is_ess_surj.
```

Theorem 8.5. *For any precategory A, there is a category $\widehat{A}$ and a weak equivalence $A \to \widehat{A}$.*

Proof. The hom-sets of A must lie in some universe Type, so that A is locally small with respect to that universe. Write $\underline{\mathsf{Set}}$ for the category of sets in Type, and let $\widehat{A}_0 :\equiv \left\{ F : \underline{\mathsf{Set}}^{A^{\mathrm{op}}} \;\middle|\; \left\| \sum(a : A), (\mathbf{y}a \cong F) \right\| \right\}$, with hom-sets inherited from $\underline{\mathsf{Set}}^{A^{\mathrm{op}}}$. In other words, $\widehat{A}$ is the full subcategory of $\underline{\mathsf{Set}}^{A^{\mathrm{op}}}$ determined by the functors that are *merely representable*. Then the inclusion $\widehat{A} \to \underline{\mathsf{Set}}^{A^{\mathrm{op}}}$ is fully faithful and a monomorphism on objects. Since $\underline{\mathsf{Set}}^{A^{\mathrm{op}}}$ is a category (by Theorem 4.5, since $\underline{\mathsf{Set}}$ is a category by univalence), $\widehat{A}$ is also a category.

Let $A \to \widehat{A}$ be the Yoneda embedding. This is fully faithful by Corollary 7.6, and essentially surjective by definition of $\widehat{A}_0$. Thus it is a weak equivalence. $\square$

What’s the catch?

```
lemma succ2_lemma : ∀ m, succ(m) * 2 = succ(succ(m*2)) :=
begin
  assume m,
  induction m with m' ih,
  refl,
  rewrite mult_comm,
  rewrite mul,
  dsimp[add],
  rewrite mult_comm,
end

lemma double_lemma : ∀ m : ℕ, m + m = m*2 :=
begin
  assume m,
  induction m with m ih,
  refl,
  dsimp[add,mul],
  rewrite add_lneutr,
end

lemma div2_lemma : ∀ m n : ℕ, div2(n + m*2) = m + (div2 n) :=
begin
  assume m n,
  induction m with m ih,
  rewrite add_lneutr,
```

Lots of math out there. . .

Was sind und was sollen Homotopy Type Theory?

- Founding observation: homotopy theory and type theory are secretly the same!
- The common language, HoTT, is actually good for expressing all kinds of mathematics in a way that's amenable to formalization
- Univalent Foundations: do *all* of math using the language of HoTT

Next Time: 3 Perspectives on HoTT

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Music:

“*Wholesome*” and “*Fluidscape*”

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