Elementary Category Theory

Theory of the Category of Sets

The Heart and Soul of Modern Mathematics

Set Theory

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century
- Dominant foundational framework for mathematics

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century
- Dominant foundational framework for mathematics
- Formally axiomatized

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century
- Dominant foundational framework for mathematics
- Formally axiomatized
  - Zermelo-Frankel axioms ($\pm$ some axioms)

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century
- Dominant foundational framework for mathematics
- Formally axiomatized
  - Zermelo-Frankel axioms ($\pm$ some axioms)

Category Theory

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century
- Dominant foundational framework for mathematics
- Formally axiomatized
  - Zermelo-Frankel axioms ($\pm$ some axioms)

Category Theory

- Emerged in the mid 20th-century

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century
- Dominant foundational framework for mathematics
- Formally axiomatized
  - Zermelo-Frankel axioms ($\pm$ some axioms)

Category Theory

- Emerged in the mid 20th-century
- Provides a different language for studying mathematical structures

Set Theory and Category Theory

Set Theory

- Primarily originated in 19th century, underwent substantial development in 20th century
- Dominant foundational framework for mathematics
- Formally axiomatized
  - Zermelo-Frankel axioms ($\pm$ some axioms)

Category Theory

- Emerged in the mid 20th-century
- Provides a different language for studying mathematical structures
- Can also serve as a foundational framework

The Theory of the Category of Sets

Lawvere's **Elementary Theory of the Category of Sets** (1964) proposes that we study the *category of sets*, i.e. use the tools & mindset of category theory to do set theory.

The Theory of the Category of Sets

Lawvere's **Elementary Theory of the Category of Sets** (1964) proposes that we study the *category of sets*, i.e. use the tools & mindset of category theory to do set theory.

This video: Explain the basic theory of sets

The Theory of the Category of Sets

Lawvere's **Elementary Theory of the Category of Sets** (1964) proposes that we study the *category of sets*, i.e. use the tools & mindset of category theory to do set theory.

This video: Explain the basic theory of sets – *as a category*, but without any explicit category theory language

The Theory of the Category of Sets

Lawvere's **Elementary Theory of the Category of Sets** (1964) proposes that we study the *category of sets*, i.e. use the tools & mindset of category theory to do set theory.

This video: Explain the basic theory of sets – *as a category*, but without any explicit category theory language
Next video: Introduce category theory and "officially" define the category of sets

The Theory of the Category of Sets

Lawvere's **Elementary Theory of the Category of Sets** (1964) proposes that we study the *category of sets*, i.e. use the tools & mindset of category theory to do set theory.

This video: Explain the basic theory of sets – *as a category*, but without any explicit category theory language

Next video: Introduce category theory and "officially" define the category of sets

Future videos: Study other interesting categories, define abstract category-theoretic "structure", build up the basics of category theory

0 The Universe of Sets

Sets

Sets

$\{0, 5, 7, 4\}$ is a set

Sets

$\{0, 5, 7, 4\}$ is a set

$0 \in \{0, 5, 7, 4\}$

Sets

$\{0, 5, 7, 4\}$ is a set

$0 \in \{0, 5, 7, 4\}$

$7 \in \{0, 5, 7, 4\}$

Sets

$\{0, 5, 7, 4\}$ is a set

$0 \in \{0, 5, 7, 4\}$

$7 \in \{0, 5, 7, 4\}$

$3 \notin \{0, 5, 7, 4\}$

Sets

$\{0, 5, 7, 4\}$ is a set

$0 \in \{0, 5, 7, 4\}$

$7 \in \{0, 5, 7, 4\}$

$3 \notin \{0, 5, 7, 4\}$

$$\{0, 5, 7, 4\} = \{4, 5, 0, 7\}$$

Sets

$\{0, 5, 7, 4\}$ is a set
$0 \in \{0, 5, 7, 4\}$
$7 \in \{0, 5, 7, 4\}$
$3 \notin \{0, 5, 7, 4\}$

$\{0, 5, 7, 4\} = \{4, 5, 0, 7\}$
$\{0, 5, 7, 4\} = \{4, 5, 5, 5, 0, 7, 5\}$

Sets

Informally, a **set** is a collection of *elements* (which could be anything, including other sets)

Sets

Informally, a **set** is a collection of *elements* (which could be anything, including other sets)

A **set** X is defined by the **elements** which are **in** X.

Sets

Informally, a **set** is a collection of *elements* (which could be anything, including other sets)

A **set** X is defined by the **elements** which are **in** X. We write $x \in X$ to mean that x is in X, and $x \notin X$ to mean that x is not in X.

Sets

Informally, a **set** is a collection of *elements* (which could be anything, including other sets)

A **set** X is defined by the **elements** which are **in** X. We write $x \in X$ to mean that x is in X, and $x \notin X$ to mean that x is not in X. If two sets have the exact same elements ($x \in X \iff x \in Y$), they are the same ($X = Y$).

Functions

Functions

A **function** f is given by:

Functions

A **function** f is given by:

- A set $\text{dom}(f)$, called the **domain** of f

Functions

A **function** f is given by:

- A set $\mathrm{dom}(f)$, called the **domain** of f
- A set $\mathrm{cod}(f)$, called the **codomain** of f

Functions

A **function** f is given by:

- A set $\text{dom}(f)$, called the **domain** of f
- A set $\text{cod}(f)$, called the **codomain** of f
- An assignment of each element $x \in \text{dom}(f)$ to an element $f(x) \in \text{cod}(f)$

Functions

A **function** f is given by:

- A set $\mathrm{dom}(f)$, called the **domain** of f
- A set $\mathrm{cod}(f)$, called the **codomain** of f
- An assignment of each element $x \in \mathrm{dom}(f)$ to an element $f(x) \in \mathrm{cod}(f)$

We'll write $f : X \to Y$ to indicate that f is a function whose domain is X and whose codomain is Y.

Functions

A **function** f is given by:

- A set $\text{dom}(f)$, called the **domain** of f
- A set $\text{cod}(f)$, called the **codomain** of f
- An assignment of each element $x \in \text{dom}(f)$ to an element $f(x) \in \text{cod}(f)$

We'll write $f : X \to Y$ to indicate that f is a function whose domain is X and whose codomain is Y. We'll sometimes write $f : x \mapsto y$ to mean that $y = f(x)$.

Lambda Notation

To easily refer to & define functions, we'll make use of **λ-notation**:

Lambda Notation

To easily refer to & define functions, we'll make use of **λ-notation**: if we define

$$f = (\lambda x.e(x)) : X \to Y$$

then, for any $z \in X$, $f(z)$ is obtained by "evaluating" the "expression" $e(z)$.

Lambda Practice

$(\lambda x.x + x) : \mathbb{R} \to \mathbb{R}$

Lambda Practice

$(\lambda x.x + x) : \mathbb{R} \to \mathbb{R}$

$(\lambda x.x + x)(4)$

Lambda Practice

$$(\lambda x.x + x) : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned}&(\lambda x.x + x)(4)\\ =\quad & 4 + 4\end{aligned}$$

Lambda Practice

$$(\lambda x.x + x) : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned}
&(\lambda x.x + x)(4) \\
= \quad & 4 + 4 \\
= \quad & 8
\end{aligned}$$

Lambda Practice

$$(\lambda x.x + x) : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned}
&(\lambda x.x + x)(4) \\
= &\quad 4 + 4 \\
= &\quad 8
\end{aligned}$$

$$(\lambda x.x + x)(2.1)$$

Lambda Practice

$$(\lambda x.x + x) : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned}
&(\lambda x.x + x)(4) \\
= &\quad 4 + 4 \\
= &\quad 8 \\
&(\lambda x.x + x)(2.1) \\
= &\quad 2.1 + 2.1
\end{aligned}$$

Lambda Practice

$$(\lambda x.x + x) : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned}
&(\lambda x.x + x)(4) \\
= \quad & 4 + 4 \\
= \quad & 8 \\
&(\lambda x.x + x)(2.1) \\
= \quad & 2.1 + 2.1 \\
= \quad & 4.2
\end{aligned}$$

Lambda Practice

$$(\lambda(x, y).\sqrt{x^2 + y^2}) : \mathbb{R}^2 \to [0, \infty)$$

Lambda Practice

$$(\lambda(x,y).\sqrt{x^2+y^2}) : \mathbb{R}^2 \to [0,\infty)$$
$$(\lambda(x,y).\sqrt{x^2+y^2})(3,4)$$

Lambda Practice

$$(\lambda(x,y).\sqrt{x^2+y^2}) : \mathbb{R}^2 \to [0,\infty)$$

$$\begin{aligned}
&(\lambda(x,y).\sqrt{x^2+y^2})(3,4) \\
= &\qquad \sqrt{3^2+4^2}
\end{aligned}$$

Lambda Practice

$$(\lambda(x,y).\sqrt{x^2+y^2}) : \mathbb{R}^2 \to [0,\infty)$$

$$\begin{aligned}
& (\lambda(x,y).\sqrt{x^2+y^2})(3,4) \\
= \quad & \sqrt{3^2+4^2} \\
= \quad & 5
\end{aligned}$$

Lambda Practice

$(\lambda n.\ \text{if } n \text{ is even then } 1 \text{ else } 0) : \mathbb{N} \to \{0, 1\}$

Lambda Practice

$(\lambda n.\ \text{if } n \text{ is even then } 1 \text{ else } 0) : \mathbb{N} \to \{0, 1\}$
$(\lambda n.\ \text{if } n \text{ is even then } 1 \text{ else } 0)(4)$

Lambda Practice

$(\lambda n.\ \text{if } n \text{ is even then } 1 \text{ else } 0) : \mathbb{N} \to \{0, 1\}$

$(\lambda n.\ \text{if } n \text{ is even then } 1 \text{ else } 0)(4)$

$= \quad \text{if } 4 \text{ is even then } 1 \text{ else } 0$

Lambda Practice

$$(\lambda n.\ \text{if } n \text{ is even then } 1 \text{ else } 0) : \mathbb{N} \to \{0, 1\}$$

$$\begin{aligned}
&(\lambda n.\ \text{if } n \text{ is even then } 1 \text{ else } 0)(4) \\
= \quad & \text{if } 4 \text{ is even then } 1 \text{ else } 0 \\
= \quad & 1
\end{aligned}$$

Lambda Practice

$(\lambda n.\ \text{if } n \text{ is even then 1 else 0}) : \mathbb{N} \to \{0, 1\}$

$(\lambda n.\ \text{if } n \text{ is even then 1 else 0})(3)$

$= \quad \text{if 3 is even then 1 else 0}$

$= \quad 0$

$(\lambda n.\ \text{if } n \text{ is even then 1 else 0})(0)$

$= \quad \text{if 0 is even then 1 else 0}$

$= \quad 1$

Function Extensionality

FunExt For $f, g : X \to Y$, if $f(x) = g(x)$ for all $x \in X$, then $f = g$.

Function Extensionality

FunExt For $f, g : X \to Y$, if $f(x) = g(x)$ for all $x \in X$, then $f = g$.

Check Your Understanding Prove:

- $f = \lambda x. f(x)$
- $(\lambda x. x + x) = (\lambda x. 2x)$

Composition

If $f : X \to Y$ and $g : Y \to Z$ (i.e. $\text{cod}(f) = \text{dom}(g)$), we can **compose** g with f:

$$g \circ f = (\lambda x.g(f(x))) : X \to Z.$$

In words: $g \circ f$ is the function which takes an input, "does f" to it, and then "does g" on the result.

Composition

If $f : X \to Y$ and $g : Y \to Z$ (i.e. $\mathrm{cod}(f) = \mathrm{dom}(g)$), we can **compose** g with f:

$$g \circ f = (\lambda x.g(f(x))) : X \to Z.$$

In words: $g \circ f$ is the function which takes an input, "does f" to it, and then "does g" on the result.

Check Your Understanding Verify:

- $h \circ (g \circ f) = (h \circ g) \circ f$ for any f, g, h with suitable (co)domains
- $$(\lambda x.x) = \left(\lambda x.\frac{x}{2}\right) \circ (\lambda x.2x)$$

Identities

For each set X, there is a special function called the **identity** on X:

$$\mathrm{id}_X = \lambda x.x : X \to X$$

Identities

For each set X, there is a special function called the **identity** on X:

$$\mathrm{id}_X = \lambda x.x : X \to X$$

Check Your Understanding Verify:

- For all $f : X \to Y$, $f \circ \mathrm{id}_X = f$
- For all $f : X \to Y$, $\mathrm{id}_Y \circ f = f$
- (More difficult) Show that if $e : Y \to Y$ is such that $g \circ e = g$ for all $g : Y \to Z$ and $e \circ f = f$ for all $f : X \to Y$, then it must be the case that $e = \mathrm{id}_Y$.

1 Some Basic Concepts of Set Theory

The Empty Set

Central to the theory of sets is the **empty set**, $\emptyset$, which is the set with *no elements*.

The Empty Set

Central to the theory of sets is the **empty set**, $\emptyset$, which is the set with *no elements*. We'll characterize $\emptyset$ as follows:

The Empty Set

Central to the theory of sets is the **empty set**, $\emptyset$, which is the set with *no elements*. We'll characterize $\emptyset$ as follows:

A set E satisfies UMP-Empty iff for every set A, there exists a unique function $E \to A$.

The Empty Set

Central to the theory of sets is the **empty set**, $\emptyset$, which is the set with *no elements*. We'll characterize $\emptyset$ as follows:

A set E satisfies UMP-Empty iff for every set A, there exists a unique function $E \to A$.

$\emptyset$ satisfies UMP-Empty:

- There is a function $f : \emptyset \to A$ for any set A: the definition of "function" is vacuously satisfied.

The Empty Set

Central to the theory of sets is the **empty set**, $\emptyset$, which is the set with *no elements*. We'll characterize $\emptyset$ as follows:

A set E satisfies UMP-Empty iff for every set A, there exists a unique function $E \to A$.

$\emptyset$ satisfies UMP-Empty:

- There is a function $f : \emptyset \to A$ for any set A: the definition of "function" is vacuously satisfied.
- This function is unique: if $f, g : \emptyset \to A$, then it vacuously holds that $f(x) = g(x)$ for all $x \in \emptyset$, so $f = g$ by FunExt.

Uniqueness of The Empty Set

Suppose E satisfies UMP-Empty.

Uniqueness of The Empty Set

Suppose E satisfies UMP-Empty.

If there were any element $x \in E$, then there would be at least two functions $E \to \{0, 1\}$: one sending $x \mapsto 0$ and the other sending $x \mapsto 1$.

Uniqueness of The Empty Set

Suppose E satisfies UMP-Empty.

If there were any element $x \in E$, then there would be at least two functions $E \to \{0, 1\}$: one sending $x \mapsto 0$ and the other sending $x \mapsto 1$. This contradicts the "uniqueness" requirement of UMP-Empty.

Uniqueness of The Empty Set

Suppose E satisfies UMP-Empty.

If there were any element $x \in E$, then there would be at least two functions $E \to \{0, 1\}$: one sending $x \mapsto 0$ and the other sending $x \mapsto 1$. This contradicts the "uniqueness" requirement of UMP-Empty. So there are no $x \in E$.

Uniqueness of The Empty Set

Suppose E satisfies UMP-Empty.

If there were any element $x \in E$, then there would be at least two functions $E \to \{0, 1\}$: one sending $x \mapsto 0$ and the other sending $x \mapsto 1$. This contradicts the "uniqueness" requirement of UMP-Empty. So there are no $x \in E$. So E must be $\emptyset$.

Uniqueness of The Empty Set

Suppose E satisfies UMP-Empty.

If there were any element $x \in E$, then there would be at least two functions $E \to \{0, 1\}$: one sending $x \mapsto 0$ and the other sending $x \mapsto 1$. This contradicts the "uniqueness" requirement of UMP-Empty. So there are no $x \in E$. So E must be $\emptyset$.

So, in conclusion,

$$E \text{ satisfies } \boxed{\text{UMP-Empty}} \iff E = \emptyset.$$

Singletons

A set T satisfies UMP-Singleton iff for every set A, there exists a unique function $!_A : A \to T$. Such a set T is called a **singleton**.

Singletons

A set T satisfies UMP-Singleton iff for every set A, there exists a unique function $!_A : A \to T$. Such a set T is called a **singleton**.

Prop. 1 If $t, t' \in T$ for some singleton T, then $t = t'$.

Singletons

A set T satisfies **UMP-Singleton** iff for every set A, there exists a unique function $!_A : A \to T$. Such a set T is called a **singleton**.

Prop. 1 If $t, t' \in T$ for some singleton T, then $t = t'$.

Prop. 2 If T is a singleton, then there's some $t \in T$.

Singletons

A set T satisfies UMP-Singleton iff for every set A, there exists a unique function $!_A : A \to T$. Such a set T is called a **singleton**.

Prop. 1 If $t, t' \in T$ for some singleton T, then $t = t'$.
Prop. 2 If T is a singleton, then there's some $t \in T$.

I.e. every singleton T has exactly one element (hence the name).

Singletons and Elements

Elements

Let $\mathbf{1}$ be the singleton $\{0\}$, and X any set. There is a correspondence between elements of X and functions $\mathbf{1} \to X$:

$$x_0 \in X \qquad \leftrightsquigarrow \qquad (\lambda 0.x_0) : \mathbf{1} \to X$$

Elements

Let $\mathbf{1}$ be the singleton $\{0\}$, and X any set. There is a correspondence between elements of X and functions $\mathbf{1} \to X$:

$$x_0 \in X \quad \leftrightsquigarrow \quad (\lambda 0.x_0) : \mathbf{1} \to X$$

$$x = x' \text{ (as elements of } X) \qquad \Longleftrightarrow \qquad x = x' \text{ (as functions } \mathbf{1} \to X)$$

Elements

Let $\mathbf{1}$ be the singleton $\{0\}$, and X any set. There is a correspondence between elements of X and functions $\mathbf{1} \to X$:

$$x_0 \in X \quad \leftrightsquigarrow \quad (\lambda 0.x_0) : \mathbf{1} \to X$$

$$x = x' \text{ (as elements of } X) \quad \Longleftrightarrow \quad x = x' \text{ (as functions } \mathbf{1} \to X)$$

If T satisfies UMP-Singleton, then there exists a unique function $!_{\mathbf{1}} : \mathbf{1} \to T$, i.e. there is a unique element of T.

Function Extensionality, Revisited

$f : X \to Y$, $x \in X$ (i.e. $x : \mathbf{1} \to X$).

$$f(x) \in Y \quad \leftrightsquigarrow \quad (f \circ x) : \mathbf{1} \to Y$$

Function Extensionality, Revisited

$f : X \to Y$, $x \in X$ (i.e. $x : \mathbf{1} \to X$).

$$f(x) \in Y \quad \leftrightsquigarrow \quad (f \circ x) : \mathbf{1} \to Y$$

FunExt If $f, f' : X \to Y$ are distinct functions (i.e. $f \neq f'$), then there is some $x : \mathbf{1} \to X$ such that

$$f \circ x \neq f' \circ x.$$

A function $f : X \to Y$ of the form $\lambda x.y_0$ for some fixed $y_0 \in Y$ is called a **constant** function.

A function $f : X \to Y$ of the form $\lambda x.y_0$ for some fixed $y_0 \in Y$ is called a **constant** function. Equivalently: $f : X \to Y$ is constant iff $f(x) = f(x')$ for all $x, x' \in X$.

A function $f : X \to Y$ of the form $\lambda x.y_0$ for some fixed $y_0 \in Y$ is called a **constant** function. Equivalently: $f : X \to Y$ is constant iff $f(x) = f(x')$ for all $x, x' \in X$.

Prop. 3 A function $f : X \to Y$ is constant if and only if $f = h \circ g$ for some $g : X \to \mathbf{1}$ and $h : \mathbf{1} \to Y$.

Bijections

A function $f : X \to Y$ is called a **bijection** if it is (left- and right-)*invertible*: there exists some $f' : Y \to X$ such that

$$f \circ f' = \mathrm{id}_Y \qquad \text{and} \qquad f' \circ f = \mathrm{id}_X .$$

f' is called the *inverse* of f.

Bijections

A function $f : X \to Y$ is called a **bijection** if it is (left- and right-)*invertible*: there exists some $f' : Y \to X$ such that

$$f \circ f' = \mathrm{id}_Y \qquad \text{and} \qquad f' \circ f = \mathrm{id}_X.$$

f' is called the *inverse* of f.

Check Your Understanding Verify:

- Compositions of bijections are bijections: if $f : X \to Y$ and $g : Y \to Z$ are bijections, so too is $g \circ f$
- Identity functions are bijections

Injectivity

Injections

$f : X \to Y$ satisfies UMP-Inj iff for all $d, e : W \to X$, $f \circ d = f \circ e$ implies $d = e$. Such an f is called a **injection** (adjective: *injective*).

Injections

$f : X \to Y$ satisfies UMP-Inj iff for all $d, e : W \to X$, $f \circ d = f \circ e$ implies $d = e$. Such an f is called a **injection** (adjective: *injective*).

Consider $W = \mathbf{1}$. Then UMP-Inj says $f(x) = f(x')$ implies $x = x'$ for all elements x, x' of X.

Injections

$f : X \to Y$ satisfies UMP-Inj iff for all $d, e : W \to X$, $f \circ d = f \circ e$ implies $d = e$. Such an f is called a **injection** (adjective: *injective*).

Consider $W = \mathbf{1}$. Then UMP-Inj says $f(x) = f(x')$ implies $x = x'$ for all elements x, x' of X.

Check Your Understanding Prove that if f satisfies UMP-Inj for just $W = \mathbf{1}$, then f satisfies UMP-Inj for all W.

Injectivity Implies Left-Invertibility

Prop. 4 $f : X \to Y$ is injective if and only if there exists $f' : Y \to X$ such that $f' \circ f = \mathrm{id}_X$ (f' is a *left inverse* for f).

Injectivity Implies Left-Invertibility

Prop. 4 $f : X \to Y$ is injective if and only if there exists $f' : Y \to X$ such that $f' \circ f = \mathrm{id}_X$ (f' is a *left inverse* for f).

Equivalent definitions of "injective":

- f satisfies **UMP-Inj**
- $f(x) = f(x')$ implies $x = x'$ for all $x, x' \in X$
- f has a *left inverse* $f' : Y \to X$

Left Invertibility Doesn't Imply Bijectivity

$f : X \to Y$ is **surjective**:

- For all $y \in Y$, there exists $x \in X$ such that

$$f(x) = y$$

$f : X \to Y$ is **surjective**:

- For all $y \in Y$, there exists $x \in X$ such that
$$f(x) = y$$
- Every element $y : \mathbf{1} \to Y$ factors as $f \circ x$ for some element $x : \mathbf{1} \to X$

$f : X \to Y$ is **surjective**:

- For all $y \in Y$, there exists $x \in X$ such that
$$f(x) = y$$
- Every element $y : \mathbf{1} \to Y$ factors as $f \circ x$ for some element $x : \mathbf{1} \to X$
- f has a *right inverse*: a function $f' : Y \to X$ such that $f \circ f' = \mathrm{id}_Y$

$f : X \to Y$ is **surjective**:

- For all $y \in Y$, there exists $x \in X$ such that
$$f(x) = y$$
- Every element $y : \mathbf{1} \to Y$ factors as $f \circ x$ for some element $x : \mathbf{1} \to X$
- f has a *right inverse*: a function $f' : Y \to X$ such that $f \circ f' = \mathrm{id}_Y$
- $f : X \to Y$ satisfies UMP-Surj: for all $g, h : Y \to Z$, $g \circ f = h \circ f$ implies $g = h$

$f : X \to Y$ is **surjective**:

- For all $y \in Y$, there exists $x \in X$ such that
$$f(x) = y$$
- Every element $y : \mathbf{1} \to Y$ factors as $f \circ x$ for some element $x : \mathbf{1} \to X$
- f has a *right inverse*: a function $f' : Y \to X$ such that $f \circ f' = \mathrm{id}_Y$
- $f : X \to Y$ satisfies UMP-Surj: for all $g, h : Y \to Z$, $g \circ f = h \circ f$ implies $g = h$

Prop. 5 A function $f : X \to Y$ is a bijection iff it is both an injection and a surjection

$f : X \to Y$ is **surjective**:

- For all $y \in Y$, there exists $x \in X$ such that
$$f(x) = y$$
- Every element $y : \mathbf{1} \to Y$ factors as $f \circ x$ for some element $x : \mathbf{1} \to X$
- f has a *right inverse*: a function $f' : Y \to X$ such that $f \circ f' = \mathrm{id}_Y$
- $f : X \to Y$ satisfies UMP-Surj: for all $g, h : Y \to Z$, $g \circ f = h \circ f$ implies $g = h$

Prop. 5 A function $f : X \to Y$ is a bijection iff it is both an injection and a surjection

Thm. (Cantor-Bernstein) If there exist injections $X \to Y$ and $Y \to X$, then there exists a bijection $X \rightleftarrows Y$

Check Your Understanding

(Easier)

- Prove that $T = \emptyset$ does not satisfy UMP-Singleton
- Use UMP-Surj to prove that the composition of two surjections is surjective
- Use UMP-Inj to prove that the composition of two injections is injective
- Prove that the unique function $\emptyset \to X$ is always injective
- Find an X such that the unique function $X \to \mathbf{1}$ is *not* surjective

Check Your Understanding

(More difficult)

- Write out the full argument for the claim: if T satisfies UMP-singleton, then there is exactly one element $t \in T$.
- Prove Prop. 4: that f is injective in the sense of

$$f(x) = f(x') \implies x = x' \qquad \text{for all } x, x' \in X$$

  if and only if f has a left inverse.
- Prove that the four definitions of 'surjective' are all equivalent
- Prove Prop. 5

2 Constructions of Sets

Products

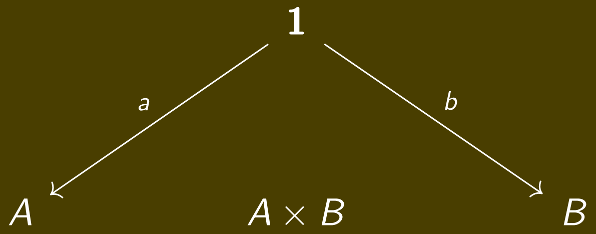
For any sets A, B, we can form their **Cartesian product** $A \times B$, whose elements are pairs:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

Products

For any sets A, B, we can form their **Cartesian product** $A \times B$, whose elements are pairs:

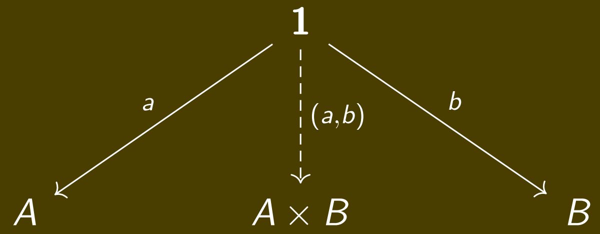
$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$



Products

For any sets A, B, we can form their **Cartesian product** $A \times B$, whose elements are pairs:

$$A \times B = \{(a, b) \mid a \in A, \; b \in B\}.$$



UMP-Prod

UMP-Prod

$$\mathrm{pr}_1 = \lambda(a, b).a$$

UMP-Prod

$$\mathrm{pr}_1 = \lambda(a, b).a$$

- $$\mathrm{pr}_1 : A \times B \to A$$

UMP-Prod

$\text{pr}_1 = \lambda(a, b).a$

- $\text{pr}_1 : A \times B \to A$

$\text{pr}_2 = \lambda(a, b).b$

UMP-Prod

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

UMP-Prod

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

Given $f : Z \to A, g : Z \to B$

UMP-Prod

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

Given $f : Z \to A, \; g : Z \to B$

$\langle f, g \rangle = \lambda z.(f(z), g(z))$

UMP-Prod

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

Given $f : Z \to A,\ g : Z \to B$

$\langle f, g \rangle = \lambda z.(f(z), g(z))$

- $\langle f, g \rangle : Z \to A \times B$

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

Given $f : Z \to A,\ g : Z \to B$

$\langle f, g \rangle = \lambda z.(f(z), g(z))$

- $\langle f, g \rangle : Z \to A \times B$

Check Your Understanding

Verify:

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

Given $f : Z \to A,\ g : Z \to B$

$\langle f, g \rangle = \lambda z.(f(z), g(z))$

- $\langle f, g \rangle : Z \to A \times B$

Check Your Understanding

Verify:

- For all $f : Z \to A$ and all $g : Z \to B$,

$$\mathrm{pr}_1 \circ \langle f, g \rangle = f$$

$$\mathrm{pr}_2 \circ \langle f, g \rangle = g$$

UMP-Prod

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

Given $f : Z \to A,\ g : Z \to B$

$\langle f, g\rangle = \lambda z.(f(z), g(z))$

- $\langle f, g\rangle : Z \to A \times B$

Check Your Understanding

Verify:

- For all $f : Z \to A$ and all $g : Z \to B$,
$$\mathrm{pr}_1 \circ \langle f, g\rangle = f$$
$$\mathrm{pr}_2 \circ \langle f, g\rangle = g$$
- For all $h : Z \to A \times B$, $\langle \mathrm{pr}_1 \circ h, \mathrm{pr}_2 \circ h\rangle = h$

UMP-Prod

$\mathrm{pr}_1 = \lambda(a, b).a$

- $\mathrm{pr}_1 : A \times B \to A$

$\mathrm{pr}_2 = \lambda(a, b).b$

- $\mathrm{pr}_2 : A \times B \to B$

Given $f : Z \to A,\ g : Z \to B$

$\langle f, g\rangle = \lambda z.(f(z), g(z))$

- $\langle f, g\rangle : Z \to A \times B$

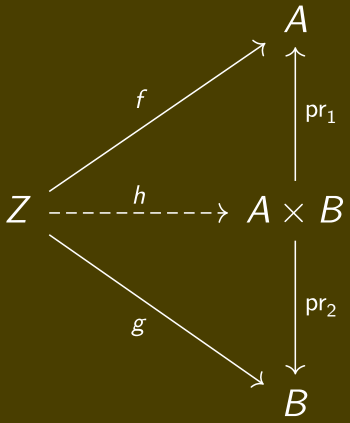
Check Your Understanding

Verify:

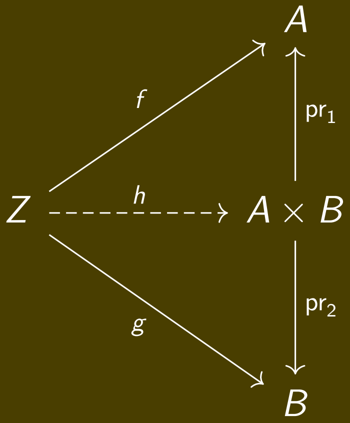
- For all $f : Z \to A$ and all $g : Z \to B$,
  $$\mathrm{pr}_1 \circ \langle f, g\rangle = f$$
  $$\mathrm{pr}_2 \circ \langle f, g\rangle = g$$
- For all $h : Z \to A \times B$,
  $$\langle \mathrm{pr}_1 \circ h, \mathrm{pr}_2 \circ h\rangle = h$$
- pr_1 and pr_2 are surjective

  (unless A or B is empty)

UMP-Prod



UMP-Prod



For each $f : Z \to A$ and $g : Z \to B$, there exists a unique $h : Z \to A \times B$ such that $\mathrm{pr}_1 \circ h = f$ and $\mathrm{pr}_2 \circ h = g$.

Exponentials

For any sets B, C, we can form their **exponential** C^B, whose elements are functions $B \to C$:

$$C^B = \{w \mid w : B \to C\}.$$

Exponentials

For any sets B, C, we can form their **exponential** C^B, whose elements are functions $B \to C$:

$$C^B = \{w \mid w : B \to C\}.$$

$$w : B \to C \quad \leftrightsquigarrow \quad w : \mathbf{1} \to C^B$$

UMP-Exp

$$\mathrm{ev} = \lambda(w, b).w(b)$$

UMP-Exp

$\mathrm{ev} = \lambda(w, b).w(b)$

- $\mathrm{ev} : C^B \times B \to C$

UMP-Exp

$\mathsf{ev} = \lambda(w, b).w(b)$

- $\mathsf{ev} : C^B \times B \to C$

Given $u : A \times B \to C$

$\text{ev} = \lambda(w, b).w(b)$
- $$\text{ev} : C^B \times B \to C$$

Given $$u : A \times B \to C$$
$\widetilde{u} = \lambda a.\lambda b.u(a, b)$

UMP-Exp

$\text{ev} = \lambda(w, b).w(b)$

- $\text{ev} : C^B \times B \to C$

Given $u : A \times B \to C$

$\widetilde{u} = \lambda a.\lambda b.u(a, b)$

- $\widetilde{u} : A \to C^B$

UMP-Exp

$\mathrm{ev} = \lambda(w, b).w(b)$

- $\mathrm{ev} : C^B \times B \to C$

Given $u : A \times B \to C$

$\widetilde{u} = \lambda a.\lambda b.u(a, b)$

- $\widetilde{u} : A \to C^B$

Given $v : A \to C^B$

UMP-Exp

$\text{ev} = \lambda(w, b).w(b)$

- $$\text{ev} : C^B \times B \to C$$

Given $u : A \times B \to C$

$\widetilde{u} = \lambda a.\lambda b.u(a, b)$

- $$\widetilde{u} : A \to C^B$$

Given $v : A \to C^B$

$\overline{v} = \lambda(a, b).(v\ a)\ b$

$\mathrm{ev} = \lambda(w, b).w(b)$

- $\mathrm{ev} : C^B \times B \to C$

Given $u : A \times B \to C$

$\widetilde{u} = \lambda a.\lambda b.u(a, b)$

- $\widetilde{u} : A \to C^B$

Given $v : A \to C^B$

$\overline{v} = \lambda(a, b).(v\ a)\ b$

- $\overline{v} : A \times B \to C$

Check Your Understanding

Verify:

UMP-Exp

$\text{ev} = \lambda(w, b).w(b)$

- $\text{ev} : C^B \times B \to C$

Given $u : A \times B \to C$

$\widetilde{u} = \lambda a.\lambda b.u(a, b)$

- $\widetilde{u} : A \to C^B$

Given $v : A \to C^B$

$\overline{v} = \lambda(a, b).(v\ a)\ b$

- $\overline{v} : A \times B \to C$

Check Your Understanding

Verify:

- For all $u : A \times B \to C$ and $v : A \to C^B$,

$$\overline{\widetilde{u}} = u \qquad \widetilde{\overline{v}} = v$$

UMP-Exp

$\mathrm{ev} = \lambda(w, b).w(b)$

- $\mathrm{ev} : C^B \times B \to C$

Given $u : A \times B \to C$

$\widetilde{u} = \lambda a.\lambda b.u(a, b)$

- $\widetilde{u} : A \to C^B$

Given $v : A \to C^B$

$\overline{v} = \lambda(a, b).(v\ a)\ b$

- $\overline{v} : A \times B \to C$

Check Your Understanding

Verify:

- For all $u : A \times B \to C$ and $v : A \to C^B$,
  $$\overline{\widetilde{u}} = u \qquad \widetilde{\overline{v}} = v$$
- For all $u : A \times B \to C$,
  $$\mathrm{ev} \circ (\lambda(a, b).(\widetilde{u}(a), b)) = u$$

Disjoint Union

Disjoint Union

Given sets A, B, we can form their **disjoint union** $A + B$, whose elements are either elements of A or elements of B:

$$A + B = \{\mathsf{inl}(a) \mid a \in A\} \cup \{\mathsf{inr}(b) \mid b \in B\}$$

Disjoint Union

Given sets A, B, we can form their **disjoint union** $A + B$, whose elements are either elements of A or elements of B:

$$A + B = \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$$

The inl and inr "tag" the elements, so every element of $A + B$ is either an element of A or B, but not both.

Disjoint Union

Given sets A, B, we can form their **disjoint union** $A + B$, whose elements are either elements of A or elements of B:

$$A + B = \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$$

The inl and inr "tag" the elements, so every element of $A + B$ is either an element of A or B, but not both.

For each $f : A \to Z$ and $g : B \to Z$, we define a unique $[f, g] : A + B \to Z$ by:

$$\begin{aligned}[f, g] = (\lambda\ & \mathrm{inl}(a).f(a) \\ & \mathrm{inr}(b).g(b))\end{aligned}$$

Finite Sets

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{1} + \mathbf{1} \quad = \quad \{\mathsf{inl}\ 0, \mathsf{inr}\ 0\}$$

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{2} \quad = \quad \mathbf{1} + \mathbf{1} \quad = \quad \{\mathsf{inl}\ 0, \mathsf{inr}\ 0\}$$

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{2} \quad = \quad \mathbf{1} + \mathbf{1} \quad = \quad \{0, 1\}$$

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{2} \quad = \quad \mathbf{1} + \mathbf{1} \quad = \quad \{0, 1\}$$
- $$\mathbf{2} + \mathbf{1} \quad = \quad \{\mathsf{inl}\ 0, \mathsf{inl}\ 1, \mathsf{inr}\ 0\}$$

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{2} \quad = \quad \mathbf{1} + \mathbf{1} \quad = \quad \{0, 1\}$$
- $$\mathbf{3} \quad = \quad \mathbf{2} + \mathbf{1} \quad = \quad \{\mathsf{inl}\ 0, \mathsf{inl}\ 1, \mathsf{inr}\ 0\}$$

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{2} \quad = \quad \mathbf{1} + \mathbf{1} \quad = \quad \{0, 1\}$$
- $$\mathbf{3} \quad = \quad \mathbf{2} + \mathbf{1} \quad = \quad \{0, 1, 2\}$$

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{2} = \mathbf{1} + \mathbf{1} = \{0, 1\}$$
- $$\mathbf{3} = \mathbf{2} + \mathbf{1} = \{0, 1, 2\}$$
- $\ldots$

Finite Sets

- We'll write $\mathbf{0}$ for $\emptyset$, because $\emptyset$ has zero elements
- As covered, $\mathbf{1} = \{0\}$ has one element
- $$\mathbf{2} = \mathbf{1} + \mathbf{1} = \{0, 1\}$$
- $$\mathbf{3} = \mathbf{2} + \mathbf{1} = \{0, 1, 2\}$$
- ...

For each finite set X, there is a unique natural number n (the *cardinality* of X) such that X is in bijection with $\mathbf{n}$

Check Your Understanding

- Show a bijection
$$A \times A \rightleftarrows A^2$$
for any set A.
- For any natural number n, show a bijection between $\mathbf{n}$ and $\mathbf{n} + \mathbf{0}$
- For any natural number n, show a bijection between $\mathbf{n}$ and $\mathbf{n} \times \mathbf{1}$

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers.

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers. Every natural number is either 0, or is of the form $m + 1$ for some $m \in \mathbb{N}$.

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers. Every natural number is either 0, or is of the form $m+1$ for some $m \in \mathbb{N}$.

$$\mathbf{1} \xrightarrow{\;0\;} \mathbb{N} \circlearrowleft \lambda m.m{+}1$$

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers. Every natural number is either 0, or is of the form $m+1$ for some $m \in \mathbb{N}$.

$$\mathbf{1} \xrightarrow{0} \mathbb{N} \circlearrowleft \lambda m.m{+}1$$

Rec-$\mathbb{N}$

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers. Every natural number is either 0, or is of the form $m + 1$ for some $m \in \mathbb{N}$.

$$\mathbf{1} \xrightarrow{0} \mathbb{N} \circlearrowleft \lambda m.m{+}1$$

Rec-$\mathbb{N}$

$$\text{recur} : (\mathbf{1} \to X) \to (X \to X) \to (\mathbb{N} \to X)$$

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers. Every natural number is either 0, or is of the form $m + 1$ for some $m \in \mathbb{N}$.

$$\mathbf{1} \xrightarrow{0} \mathbb{N} \circlearrowleft \lambda m.m{+}1$$

Rec-$\mathbb{N}$

$$\text{recur} : \quad X \quad \to (X \to X) \to (\mathbb{N} \to X)$$

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers. Every natural number is either 0, or is of the form $m+1$ for some $m \in \mathbb{N}$.

$$\mathbf{1} \xrightarrow{0} \mathbb{N} \circlearrowleft \lambda m.m{+}1$$

Rec-$\mathbb{N}$

$$\text{recur} : \quad X \quad \to (X \to X) \to (\mathbb{N} \to X)$$

$$\text{recur } x_0 \; g \; 0 = x_0$$

The Natural Numbers

There is a set $\mathbb{N}$ of natural numbers. Every natural number is either 0, or is of the form $m+1$ for some $m \in \mathbb{N}$.

$$\mathbf{1} \xrightarrow{0} \mathbb{N} \circlearrowleft \lambda m.m{+}1$$

Rec-$\mathbb{N}$

$$\begin{aligned} \text{recur} : \quad X \quad &\to (X \to X) \to (\mathbb{N} \to X) \\ \text{recur } x_0 \ g \ 0 &= x_0 \\ \text{recur } x_0 \ g \ (m+1) &= g(\text{recur } x_0 \ g \ m) \end{aligned}$$

Indexed Union

Suppose I is a set, and for each $i \in I$, we have a set A_i.

Indexed Union

Suppose I is a set, and for each $i \in I$, we have a set A_i. Then define:

$$\sum_{i \in I} A_i \quad = \quad \{(i, a) \mid i \in I,\ a \in A_i\}$$

Indexed Union

Suppose I is a set, and for each $i \in I$, we have a set A_i. Then define:

$$\sum_{i \in I} A_i \quad = \quad \{(i, a) \mid i \in I,\ a \in A_i\}$$

$$X + Y = \sum_{i \in \mathbf{2}} A_i \qquad \text{where } A_0 = X \text{ and } A_1 = Y$$

$$X \times Y = \sum_{x \in X} Y$$

Check Your Understanding

- Formulate a Universal Mapping Property for the (binary) disjoint union operation
- For any set X, construct a bijection

$$X \rightleftarrows \sum_{x \in X} \mathbf{1}.$$

- A *sequence* in a set X is a countably-infinite, ordered collection $\{x_i\}_{i \in \mathbb{N}}$. Describe the set of all such sequences in terms of functions.
- Prove for any sets A, B, C that there is a bijection

$$(A \times C) + (B \times C) \to (A + B) \times C$$

3 Theory of Subsets

Subsets

A set A is a **subset** of B – written $A \subseteq B$ – if every element of A is a also an element of B:

Subsets

A set A is a **subset** of B – written $A \subseteq B$ – if every element of A is a also an element of B:

$$a \in A \qquad \Longrightarrow \qquad a \in B.$$

Subsets

A set A is a **subset** of B – written $A \subseteq B$ – if every element of A is a also an element of B:

$$a \in A \qquad \Longrightarrow \qquad a \in B.$$

Example: The sets evens $= \{2n \mid n \in \mathbb{N}\}$ and odds $= \{2n + 1 \mid n \in \mathbb{N}\}$ are both subsets of $\mathbb{N}$.

Subsets

A set A is a **subset** of B – written $A \subseteq B$ – if every element of A is a also an element of B:

$$a \in A \qquad \Longrightarrow \qquad a \in B.$$

Example: The sets evens $= \{2n \mid n \in \mathbb{N}\}$ and odds $= \{2n+1 \mid n \in \mathbb{N}\}$ are both subsets of $\mathbb{N}$.

Check Your Understanding Verify:

- For all A, $\emptyset \subseteq A$
- For all A, $A \subseteq A$
- If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Two ways of capturing the “subset” relation using functions.

Two ways of capturing the "subset" relation using functions.

Inclusions

Two ways of capturing the "subset" relation using functions.

Inclusions

If $A \subseteq B$, then there is a (unique) function $i_A : A \to B$ which takes each $a \in A$ to itself:

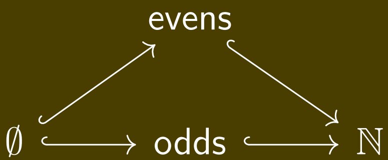
$$i_A(a) = a \in B.$$

Two ways of capturing the "subset" relation using functions.

Inclusions

If $A \subseteq B$, then there is a (unique) function $i_A : A \to B$ which takes each $a \in A$ to itself:

$$i_A(a) = a \in B.$$

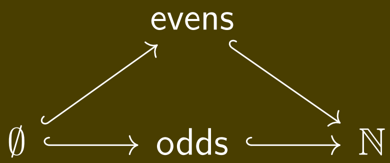


Two ways of capturing the “subset” relation using functions.

Inclusions

If $A \subseteq B$, then there is a (unique) function $i_A : A \to B$ which takes each $a \in A$ to itself:

$$i_A(a) = a \in B.$$



Characteristic Functions

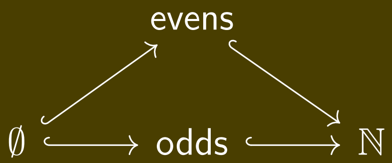
Each subset $A \subseteq B$ is associated with a unique function

Two ways of capturing the “subset” relation using functions.

Inclusions

If $A \subseteq B$, then there is a (unique) function $i_A : A \to B$ which takes each $a \in A$ to itself:

$$i_A(a) = a \in B.$$



Characteristic Functions

Each subset $A \subseteq B$ is associated with a unique function

$$\chi_A : B \to \mathbf{2}$$

given by

$$\chi_A = \lambda b.\text{if } b \in A \text{ then } 1 \text{ else } 0$$

i.e. it returns 1 on all elements of A, and 0 on all non-elements of A.

Subset Classification

For any given subset $A \subseteq B$, these two functions are related. Notice that, for any $a \in A$, $\chi_A(i_A(a)) = 1$,

Subset Classification

For any given subset $A \subseteq B$, these two functions are related. Notice that, for any $a \in A$, $\chi_A(i_A(a)) = 1$, i.e. $\chi_A \circ i_A$ is the constant function $(\lambda a.1) : A \to \mathbf{2}$.

Let $\text{true} : \mathbf{1} \to \mathbf{2}$ be the element $1 \in \mathbf{2}$. Recall that $!_X$ is the unique function $X \to \mathbf{1}$.

Subset Classification

For any given subset $A \subseteq B$, these two functions are related. Notice that, for any $a \in A$, $\chi_A(i_A(a)) = 1$, i.e. $\chi_A \circ i_A$ is the constant function $(\lambda a.1) : A \to \mathbf{2}$.

Let $\mathsf{true} : \mathbf{1} \to \mathbf{2}$ be the element $1 \in \mathbf{2}$. Recall that $!_X$ is the unique function $X \to \mathbf{1}$. Then, for any $A \subseteq B$,

$$\mathsf{true} \circ !_A \quad = \quad \chi_A \circ i_A$$

Subset Classification

For any given subset $A \subseteq B$, these two functions are related. Notice that, for any $a \in A$, $\chi_A(i_A(a)) = 1$, i.e. $\chi_A \circ i_A$ is the constant function $(\lambda a.1) : A \to \mathbf{2}$.

Let $\mathsf{true} : \mathbf{1} \to \mathbf{2}$ be the element $1 \in \mathbf{2}$. Recall that $!_X$ is the unique function $X \to \mathbf{1}$. Then, for any $A \subseteq B$,

$$\mathsf{true} \circ !_A \quad = \quad \chi_A \circ i_A$$

$$\begin{array}{ccc} A & \xrightarrow{\;!_A\;} & \mathbf{1} \\ {\scriptstyle i_A}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{true}} \\ B & \xrightarrow[\chi_A]{} & \mathbf{2} \end{array}$$

There's a (more complex) sense in which i_A and χ_A are the *optimal* functions satisfying this equation

Powersets

Since we have the correspondence identifying subsets with their characteristic functions.

$$A \subseteq B \quad \leftrightsquigarrow \quad \chi_A : B \to \mathbf{2}$$

Powersets

Since we have the correspondence identifying subsets with their characteristic functions.

$$A \subseteq B \quad \leftrightsquigarrow \quad \chi_A : B \to \mathbf{2}$$

we can define the **power set** of B – denoted $\mathcal{P}(B)$ – to be the set $\mathbf{2}^B$ of all functions $B \to \mathbf{2}$,

Powersets

Since we have the correspondence identifying subsets with their characteristic functions.

$$A \subseteq B \quad \leftrightsquigarrow \quad \chi_A : B \to \mathbf{2}$$

we can define the **power set** of B – denoted $\mathcal{P}(B)$ – to be the set $\mathbf{2}^B$ of all functions $B \to \mathbf{2}$, that is, the set of all subsets of B.

Powersets

Since we have the correspondence identifying subsets with their characteristic functions.

$$A \subseteq B \quad \leftrightsquigarrow \quad \chi_A : B \to \mathbf{2}$$

we can define the **power set** of B – denoted $\mathcal{P}(B)$ – to be the set $\mathbf{2}^B$ of all functions $B \to \mathbf{2}$, that is, the set of all subsets of B.

Thm. (Cantor) There are no surjections $X \to \mathcal{P}(X)$ (for any set X).

Comprehension

Given a set B and a function $\phi : B \to \mathbf{2}$, we'll often write

$$\{b \in B \mid \phi(b)\}$$

to mean the subset of B whose characteristic function is ϕ, i.e $\{b \in B \mid \phi(b)\}$ is the set of all those $b \in B$ such that $\phi(b) = 1$.

Comprehension

Given a set B and a function $\phi : B \to \mathbf{2}$, we'll often write

$$\{b \in B \mid \phi(b)\}$$

to mean the subset of B whose characteristic function is ϕ, i.e $\{b \in B \mid \phi(b)\}$ is the set of all those $b \in B$ such that $\phi(b) = 1$.

Example:

$$\{x \in \mathbb{R} \mid x^2 = 2\} = \left\{\sqrt{2}, -\sqrt{2}\right\}$$

Comprehension

Given a set B and a function $\phi : B \to \mathbf{2}$, we'll often write

$$\{b \in B \mid \phi(b)\}$$

to mean the subset of B whose characteristic function is ϕ, i.e $\{b \in B \mid \phi(b)\}$ is the set of all those $b \in B$ such that $\phi(b) = 1$.

Example:

$$\{x \in \mathbb{R} \mid x^2 = 2\} = \left\{\sqrt{2}, -\sqrt{2}\right\}$$

(here, ϕ is $\lambda x.$if $x^2 = 2$ then 1 else 0, and $\pm\sqrt{2}$ are the only two values $x \in \mathbb{R}$ that make $\phi(x) = 1$).

Fibers

Given a function $f : X \to Y$ and an element $y \in Y$, we can form the **fiber** of y,

$$\mathrm{fib}_f(y) = \{x \in X \mid f(x) = y\}$$

Fibers

Given a function $f : X \to Y$ and an element $y \in Y$, we can form the **fiber** of y,

$$\mathrm{fib}_f(y) = \{x \in X \mid f(x) = y\}$$

Check Your Understanding

- For $f = (\lambda x.x^2) : \mathbb{R} \to \mathbb{R}$, calculate $\mathrm{fib}_f(y)$ for $y = 2$, $y = 0$, and $y = -1$
- For all $f : X \to Y$, construct a bijection $X \rightleftarrows \sum_{y \in Y} \mathrm{fib}_f(y)$

UMP-Fib

Given a function $f : X \to Y$ and some fixed $y_0 \in Y$, let $W = \text{fib}_f(y_0)$.

UMP-Fib

Given a function $f : X \to Y$ and some fixed $y_0 \in Y$, let $W = \mathrm{fib}_f(y_0)$. Observe that the inclusion function $i_W : W \hookrightarrow X$ satisfies

$$f \circ i_W \quad = \quad (\lambda x.y_0) \circ i_W.$$

UMP-Fib

Given a function $f : X \to Y$ and some fixed $y_0 \in Y$, let $W = \mathrm{fib}_f(y_0)$. Observe that the inclusion function $i_W : W \hookrightarrow X$ satisfies

$$f \circ i_W = (\lambda x.y_0) \circ i_W.$$

$$W \xhookrightarrow{i_W} X \underset{\lambda x.y_0}{\overset{f}{\rightrightarrows}} Y$$

UMP-Fib

Given a function $f : X \to Y$ and some fixed $y_0 \in Y$, let $W = \mathrm{fib}_f(y_0)$. Observe that the inclusion function $i_W : W \hookrightarrow X$ satisfies

$$f \circ i_W \quad = \quad (\lambda x.y_0) \circ i_W.$$

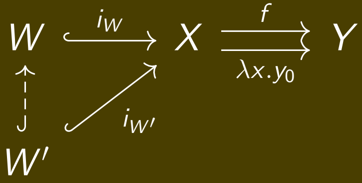
$$W \xhookrightarrow{i_W} X \underset{\lambda x.y_0}{\overset{f}{\rightrightarrows}} Y$$

Moreover, W is the *largest subset of X with this property*: if there's any other $W' \subseteq X$ such that $f \circ i_{W'} = (\lambda x.y_0) \circ i_{W'}$, then $W' \subseteq W$

UMP-Fib

Given a function $f : X \to Y$ and some fixed $y_0 \in Y$, let $W = \mathrm{fib}_f(y_0)$. Observe that the inclusion function $i_W : W \hookrightarrow X$ satisfies

$$f \circ i_W = (\lambda x.y_0) \circ i_W.$$



Moreover, W is the *largest subset of* X *with this property*: if there's any other $W' \subseteq X$ such that $f \circ i_{W'} = (\lambda x.y_0) \circ i_{W'}$, then $W' \subseteq W$

Generalization: Equalizers (UMP-Eqlzr)

Given functions $f, g : X \to Y$, define the **equalizer** of f and g to be the set $E = \{x \in X \mid f(x) = g(x)\}$.

Generalization: Equalizers (UMP-Eqlzr)

Given functions $f, g : X \to Y$, define the **equalizer** of f and g to be the set $E = \{x \in X \mid f(x) = g(x)\}$.
E has the following property:

Generalization: Equalizers (UMP-Eqlzr)

Given functions $f, g : X \to Y$, define the **equalizer** of f and g to be the set $E = \{x \in X \mid f(x) = g(x)\}$.
E has the following property:

- The inclusion function $i_E : E \hookrightarrow X$ satisfies $f \circ i_E = g \circ i_E$

Generalization: Equalizers (UMP-Eqlzr)

Given functions $f, g : X \to Y$, define the **equalizer** of f and g to be the set $E = \{x \in X \mid f(x) = g(x)\}$.
E has the following property:

- The inclusion function $i_E : E \hookrightarrow X$ satisfies $f \circ i_E = g \circ i_E$

$$E \xhookrightarrow{i_E} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Generalization: Equalizers (UMP-Eqlzr)

Given functions $f, g : X \to Y$, define the **equalizer** of f and g to be the set $E = \{x \in X \mid f(x) = g(x)\}$.

E has the following property:

- The inclusion function $i_E : E \hookrightarrow X$ satisfies $f \circ i_E = g \circ i_E$

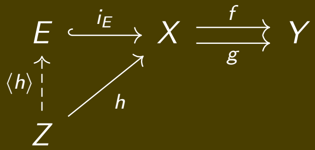
$$E \xhookrightarrow{i_E} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

- For any other set Z and any function $h : Z \to X$ such that $f \circ h = g \circ h$, there exists a unique function $\langle h \rangle : Z \to E$ such that $h = i_E \circ \langle h \rangle$.

Generalization: Equalizers (UMP-Eqlzr)

Given functions $f, g : X \to Y$, define the **equalizer** of f and g to be the set $E = \{x \in X \mid f(x) = g(x)\}$.
E has the following property:

- The inclusion function $i_E : E \hookrightarrow X$ satisfies $f \circ i_E = g \circ i_E$



- For any other set Z and any function $h : Z \to X$ such that $f \circ h = g \circ h$, there exists a unique function $\langle h \rangle : Z \to E$ such that $h = i_E \circ \langle h \rangle$.

Check Your Understanding

Given f, g, h as on the previous slide, verify:

- $f \circ i_E = g \circ i_E$

We'll define the function $\langle h \rangle : Z \to E$ demanded by UMP-Eqlzr by

$$\langle h \rangle = \lambda z.h(z)$$

Verify:

- For every $z \in Z$, $\langle h \rangle (z) \in E$
- $i_E \circ \langle h \rangle = h$

Generalization: Fiber Products

Suppose we have sets A, B, C and functions $f : A \to C$ and $g : B \to C$. Define the **fiber product** of f and g (denoted $A \times_C B$) to be the set

Generalization: Fiber Products

Suppose we have sets A, B, C and functions $f : A \to C$ and $g : B \to C$. Define the **fiber product** of f and g (denoted $A \times_C B$) to be the set

$$\{(a, b) \in A \times B \mid f(a) = g(b)\}$$

Generalization: Fiber Products

Suppose we have sets A, B, C and functions $f : A \to C$ and $g : B \to C$. Define the **fiber product** of f and g (denoted $A \times_C B$) to be the set

$$\{(a, b) \in A \times B \mid f(a) = g(b)\}$$

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\mathrm{pr}_2 = \lambda(a,b).b} & B \\ {\scriptstyle \mathrm{pr}_1 = \lambda(a,b).a}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

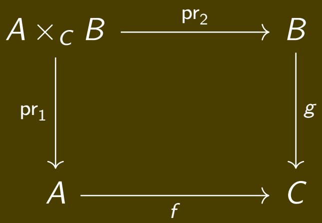
$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2$$

UMP-FibProd

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\mathrm{pr}_2} & B \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

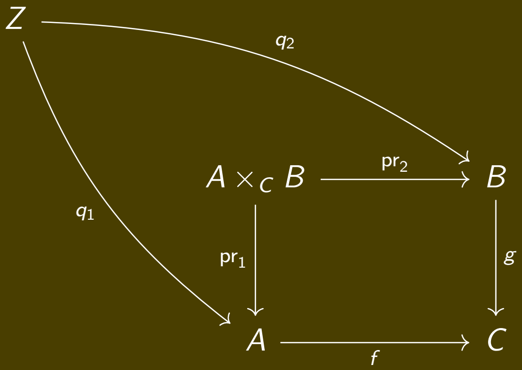
UMP-FibProd

Z



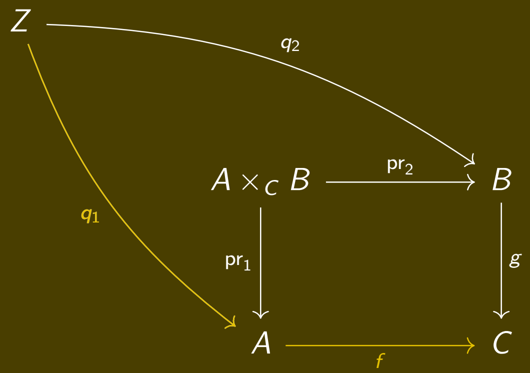
For all sets Z

UMP-FibProd



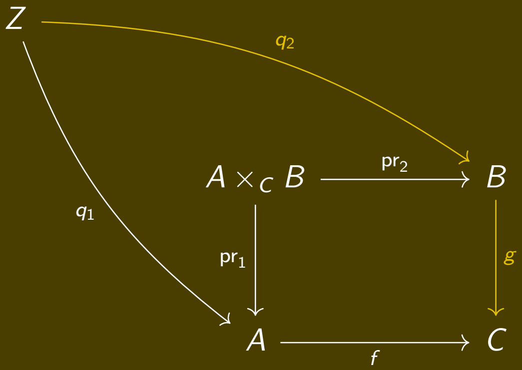
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$

UMP-FibProd



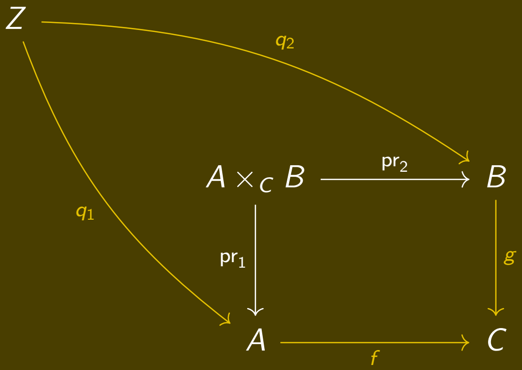
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1$

UMP-FibProd



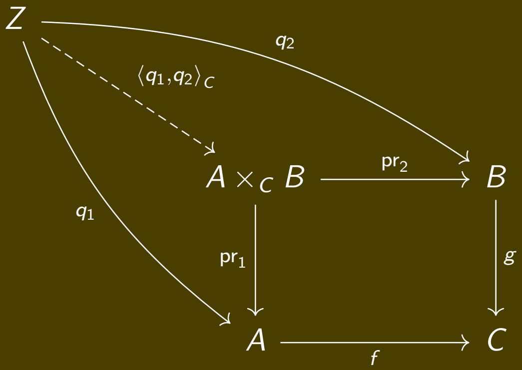
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1$ $g \circ q_2$

UMP-FibProd



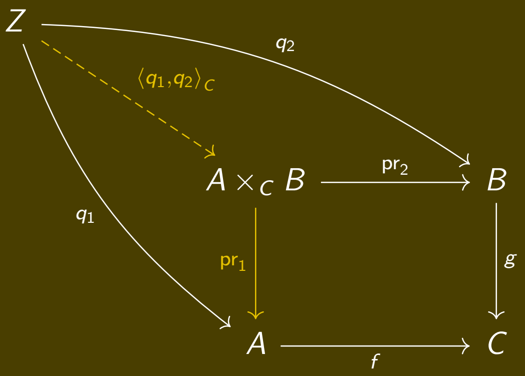
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$,

UMP-FibProd



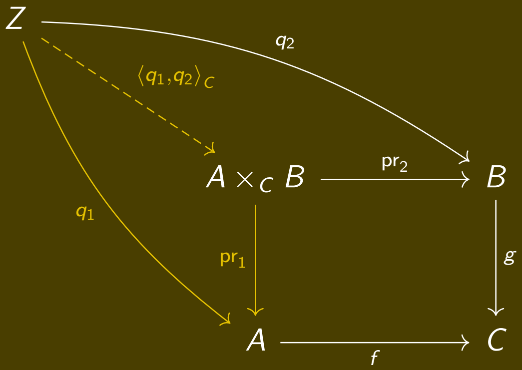
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique function $\langle q_1, q_2 \rangle_C : Z \to A \times_C B$

UMP-FibProd



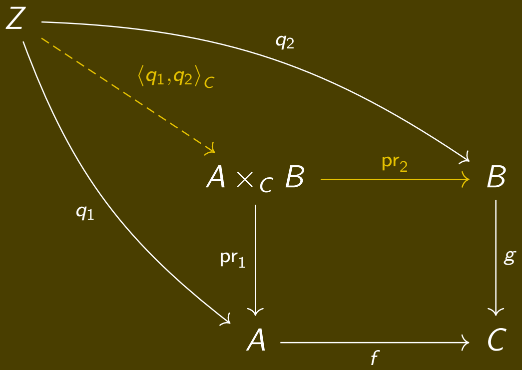
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique function $\langle q_1, q_2 \rangle_C : Z \to A \times_C B$ such that $\mathrm{pr}_1 \circ \langle q_1, q_2 \rangle_C$

UMP-FibProd



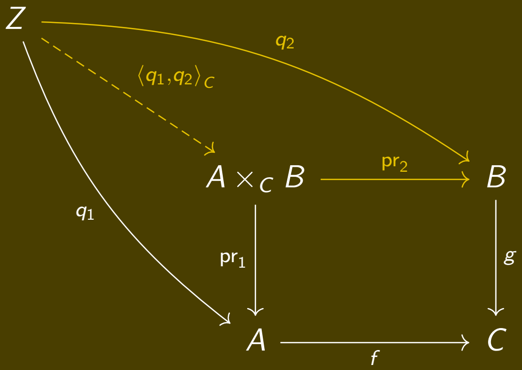
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique function $\langle q_1, q_2 \rangle_C : Z \to A \times_C B$ such that $\mathrm{pr}_1 \circ \langle q_1, q_2 \rangle_C = q_1$

UMP-FibProd



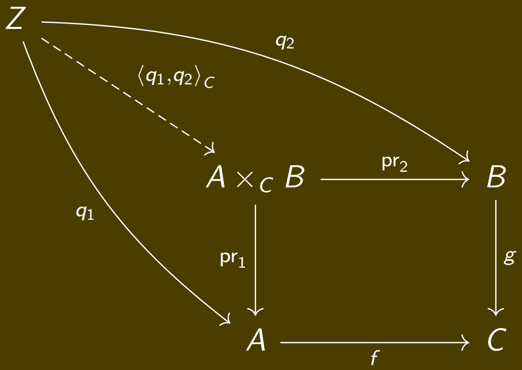
For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique function $\langle q_1, q_2 \rangle_C : Z \to A \times_C B$ such that $\mathrm{pr}_1 \circ \langle q_1, q_2 \rangle_C = q_1$ and $\mathrm{pr}_2 \circ \langle q_1, q_2 \rangle_C$

UMP-FibProd



For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique function $\langle q_1, q_2\rangle_C : Z \to A \times_C B$ such that $\mathrm{pr}_1 \circ \langle q_1, q_2\rangle_C = q_1$ and $\mathrm{pr}_2 \circ \langle q_1, q_2\rangle_C = q_2$

UMP-FibProd



For all sets Z and all functions $q_1 : Z \to A$, $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique function $\langle q_1, q_2 \rangle_C : Z \to A \times_C B$ such that $\mathrm{pr}_1 \circ \langle q_1, q_2 \rangle_C = q_1$ and $\mathrm{pr}_2 \circ \langle q_1, q_2 \rangle_C = q_2$

Check Your Understanding

- State UMP-FibProd for the case $Z = \mathbf{1}$ and interpret it as a statement describing the elements of $A \times_C B$
- For any sets A, B, find a set C and functions $f : A \to C$, $g : B \to C$ such that $A \times_C B$ is the cartesian product $A \times B$

Check Your Understanding

A square of sets and functions
$$\begin{array}{ccc} P & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$
is called a **pullback square**

if $f \circ p_1 = g \circ p_2$ and it satisfies UMP-FibProd: for any Z and any $q_1 : Z \to A$ and $q_2 : Z \to B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique $h : Z \to P$ such that $q_1 = p_1 \circ h$ and $q_2 = p_2 \circ h$.

For any subset $X \subseteq Y$, show that the square

$$\begin{array}{ccc} X & \xrightarrow{!_X} & \mathbf{1} \\ {\scriptstyle i_X}\big\downarrow & & \big\downarrow{\scriptstyle \text{true}} \\ Y & \xrightarrow[\chi_X]{} & \mathbf{2} \end{array}$$

is a pullback square.

Kernels

Consider the fiber product of $f : A \to C$ with itself:

$$\begin{array}{ccc} K = A \times_C A & \xrightarrow{k_2 = \lambda(a,a').a'} & A \\ {\scriptstyle k_1 = \lambda(a,a').a}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & C \end{array}$$

Kernels

Consider the fiber product of $f : A \to C$ with itself:

$$\begin{array}{ccc} K = A \times_C A & \xrightarrow{k_2 = \lambda(a,a').a'} & A \\ {\scriptstyle k_1 = \lambda(a,a').a}\big\downarrow & & \big\downarrow {\scriptstyle f} \\ A & \xrightarrow[f]{} & C \end{array}$$

$K \subseteq A \times A$ is the set of all pairs (a, a') such that $f(a) = f(a')$.

Kernels

Consider the fiber product of $f : A \to C$ with itself:

$$
\begin{array}{ccc}
K = A \times_C A & \xrightarrow{k_2 = \lambda(a,a').a'} & A \\
{\scriptstyle k_1 = \lambda(a,a').a}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
A & \xrightarrow[f]{} & C
\end{array}
$$

$K \subseteq A \times A$ is the set of all pairs (a, a') such that $f(a) = f(a')$. k_1, k_2 is called the **kernel pair** of f.

Equivalence Relations

The kernel $K = \{(a, a') \in A \times A \mid f(a) = f(a')\}$ satisfies three properties:

Equivalence Relations

The kernel $K = \{(a, a') \in A \times A \mid f(a) = f(a')\}$ satisfies three properties:

- Reflexivity For all $a \in A$, $(a, a) \in K$

Equivalence Relations

The kernel $K = \{(a, a') \in A \times A \mid f(a) = f(a')\}$ satisfies three properties:

- **Reflexivity** For all $a \in A$, $(a, a) \in K$
- **Symmetry** If $(a, a') \in K$, then $(a', a) \in K$

Equivalence Relations

The kernel $K = \{(a, a') \in A \times A \mid f(a) = f(a')\}$ satisfies three properties:

- **Reflexivity** For all $a \in A$, $(a, a) \in K$
- **Symmetry** If $(a, a') \in K$, then $(a', a) \in K$
- **Transitivity** If $(a, a') \in K$ and $(a', a'') \in K$, then $(a, a'') \in K$.

Equivalence Relations

The kernel $K = \{(a, a') \in A \times A \mid f(a) = f(a')\}$ satisfies three properties:

- Reflexivity For all $a \in A$, $(a, a) \in K$
- Symmetry If $(a, a') \in K$, then $(a', a) \in K$
- Transitivity If $(a, a') \in K$ and $(a', a'') \in K$, then $(a, a'') \in K$.

A subset of $A \times A$ satisfying these three conditions is called an **equivalence relation** on A.

Check Your Understanding

Verify:

- $R = A \times A$ is an equivalence relation on A
- $R = \Delta_A = \{(a, a) \mid a \in A\}$ is an equivalence relation on A
- The set $R \subseteq \mathbb{Z} \times \mathbb{Z}$ given by

$$\{(p, q) \in \mathbb{Z} \times \mathbb{Z} \mid (p - q) \text{ is a multiple of } 2\}$$

  is an equivalence relation.

Quotients

Given an equivalence relation R on A and an element $a \in A$, define the **equivalence class** of a to be the set

$$[a]_R = \{a' \in A \mid (a, a') \in R\} \subseteq A$$

Quotients

Given an equivalence relation R on A and an element $a \in A$, define the **equivalence class** of a to be the set

$$[a]_R = \{a' \in A \mid (a, a') \in R\} \subseteq A$$

Notice that $(a, a') \in R$ iff $[a]_R = [a']_R$.

Quotients

Given an equivalence relation R on A and an element $a \in A$, define the **equivalence class** of a to be the set

$$[a]_R = \{a' \in A \mid (a, a') \in R\} \subseteq A$$

Notice that $(a, a') \in R$ iff $[a]_R = [a']_R$.
Then put

$$A/R = \{[a]_R \mid a \in A\}.$$

Quotients

Given an equivalence relation R on A and an element $a \in A$, define the **equivalence class** of a to be the set

$$[a]_R = \{a' \in A \mid (a, a') \in R\} \subseteq A$$

Notice that $(a, a') \in R$ iff $[a]_R = [a']_R$.
Then put

$$A/R = \{[a]_R \mid a \in A\}.$$

And write π_R for the function $(\lambda a.[a]_R) : A \to A/R$.

Quotients

Given an equivalence relation R on A and an element $a \in A$, define the **equivalence class** of a to be the set

$$[a]_R = \{a' \in A \mid (a, a') \in R\} \subseteq A$$

Notice that $(a, a') \in R$ iff $[a]_R = [a']_R$.
Then put

$$A/R = \{[a]_R \mid a \in A\}.$$

And write π_R for the function $(\lambda a.[a]_R) : A \to A/R$.

Check Your Understanding

Prove that π_R is a surjection for any A and R.

UMP-Quot

Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.

$$R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} A \xrightarrow{\pi_R} A/R$$

UMP-Quot

Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.

$$R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} A \xrightarrow{\pi_R} A/R$$

UMP-Quot

Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.

$$R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} A \xrightarrow{\pi_R} A/R$$

- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$

UMP-Quot

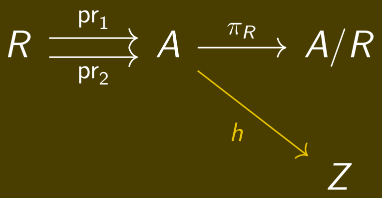
Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.

$$R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} A \xrightarrow{\pi_R} A/R$$

- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$

UMP-Quot

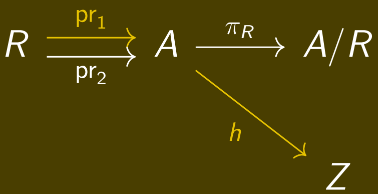
Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.



- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$
- If $h : A \to Z$

UMP-Quot

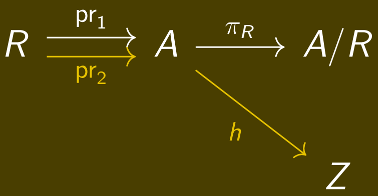
Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.



- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$
- If $h : A \to Z$ is such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$,

UMP-Quot

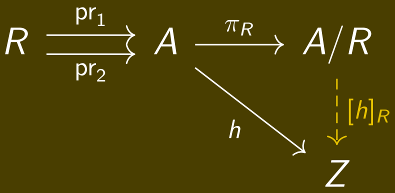
Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.



- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$
- If $h : A \to Z$ is such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$,

UMP-Quot

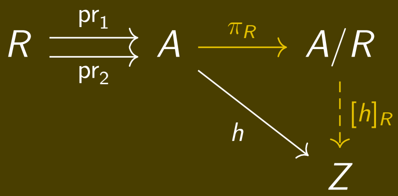
Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.



- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$
- If $h : A \to Z$ is such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$, then there exists a unique $[h] : A/R \to Z$

UMP-Quot

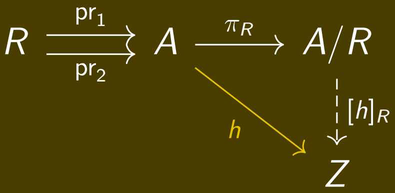
Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.



- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$
- If $h : A \to Z$ is such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$, then there exists a unique $[h] : A/R \to Z$ such that $[h] \circ \pi_R = h$.

UMP-Quot

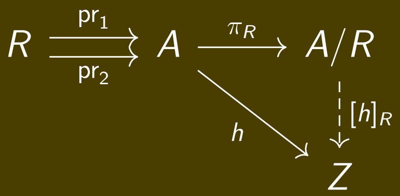
Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.



- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$
- If $h : A \to Z$ is such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$, then there exists a unique $[h] : A/R \to Z$ such that $[h] \circ \pi_R = h$.

UMP-Quot

Since $R \subseteq A \times A$, we have $\mathrm{pr}_1 = (\lambda(a, a').a) : R \to A$ and $\mathrm{pr}_2 = (\lambda(a, a').a') : R \to A$.



- $\pi_R \circ \mathrm{pr}_1 = \pi_R \circ \mathrm{pr}_2$
- If $h : A \to Z$ is such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$, then there exists a unique $[h] : A/R \to Z$ such that $[h] \circ \pi_R = h$.

Functions out of the quotient

By UMP-Quot, every function $g : A/R \to Z$ is of the form $[h]$ for some $h : A \to Z$ such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$.

Functions out of the quotient

By UMP-Quot, every function $g : A/R \to Z$ is of the form $[h]$ for some $h : A \to Z$ such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$. In particular,

$$g = [g \circ \pi_R].$$

Functions out of the quotient

By UMP-Quot, every function $g : A/R \to Z$ is of the form $[h]$ for some $h : A \to Z$ such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$. In particular,

$$g = [g \circ \pi_R].$$

Functions out of the quotient

By UMP-Quot, every function $g : A/R \to Z$ is of the form $[h]$ for some $h : A \to Z$ such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$. In particular,

$$g = [g \circ \pi_R].$$

Given $h : A \to Z$

Functions out of the quotient

By UMP-Quot, every function $g : A/R \to Z$ is of the form $[h]$ for some $h : A \to Z$ such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$. In particular,

$$g = [g \circ \pi_R].$$

Given $h : A \to Z$
$(\lambda[a]_R.h(a)) \; : \; A/R \to Z$

Functions out of the quotient

By UMP-Quot, every function $g : A/R \to Z$ is of the form $[h]$ for some $h : A \to Z$ such that $h \circ \mathrm{pr}_1 = h \circ \mathrm{pr}_2$. In particular,

$$g = [g \circ \pi_R].$$

Given $h : A \to Z$
$(\lambda[a]_R.h(a)) \ : \ A/R \to Z$
REQUIRES: if $[a]_R = [a']_R$ (i.e. $(a, a') \in R$), then $h(a) = h(a')$.

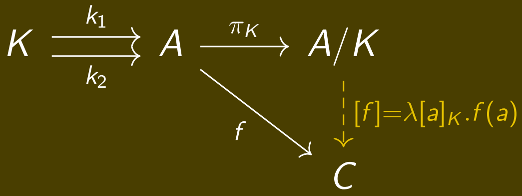
Quotienting by the Kernel

Given a function $f : A \to C$ with kernel pair $K \underset{k_2}{\overset{k_1}{\rightrightarrows}} A$, construct the quotient diagram

$$
\begin{array}{ccccc}
K & \underset{k_2}{\overset{k_1}{\rightrightarrows}} & A & \xrightarrow{\pi_K} & A/K \\
 & & & \searrow^{f} & \\
 & & & & C
\end{array}
$$

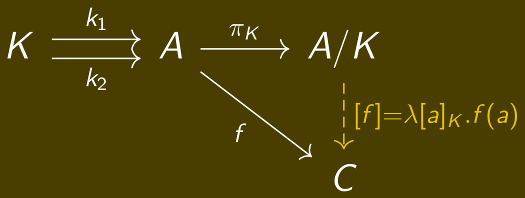
Quotienting by the Kernel

Given a function $f : A \to C$ with kernel pair $K \underset{k_2}{\overset{k_1}{\rightrightarrows}} A$, construct the quotient diagram



Quotienting by the Kernel

Given a function $f : A \to C$ with kernel pair $K \overset{k_1}{\underset{k_2}{\rightrightarrows}} A$, construct the quotient diagram



Check Your Understanding Verify that the lambda function $\lambda[a]_K.f(a)$ satisfies the requirements from the previous slide, and verify $f = (\lambda[a]_K.f(a)) \circ \pi_K$.

Images

Given $f : A \to C$ as before, define the **image** of f to be the set

$$\mathrm{im}(f) = \{c \in C \mid c = f(a) \text{ for some } a \in A\} = \{c \in C \mid \mathrm{fib}_f(c) \neq \emptyset\}$$

Images

Given $f : A \to C$ as before, define the **image** of f to be the set

$$\operatorname{im}(f) = \{c \in C \mid c = f(a) \text{ for some } a \in A\} = \{c \in C \mid \operatorname{fib}_f(c) \neq \emptyset\}$$

Prop. 6 For $f : A \to C$ with kernel K,

$$A/K = \{\operatorname{fib}_f(c) \mid c \in \operatorname{im}(f)\}$$

Images

Given $f : A \to C$ as before, define the **image** of f to be the set

$$\mathrm{im}(f) = \{c \in C \mid c = f(a) \text{ for some } a \in A\} = \{c \in C \mid \mathrm{fib}_f(c) \neq \emptyset\}$$

Prop. 6 For $f : A \to C$ with kernel K,

$$A/K = \{\mathrm{fib}_f(c) \mid c \in \mathrm{im}(f)\}$$

Prop. 7 $\mathrm{fib}_f : \mathrm{im}(f) \to A/K$ is a bijection:

$$A/K \underset{\mathrm{fib}_f}{\overset{\lambda[a]_K.f(a)}{\rightleftarrows}} \mathrm{im}(f)$$

Images

Given $f : A \to C$ as before, define the **image** of f to be the set

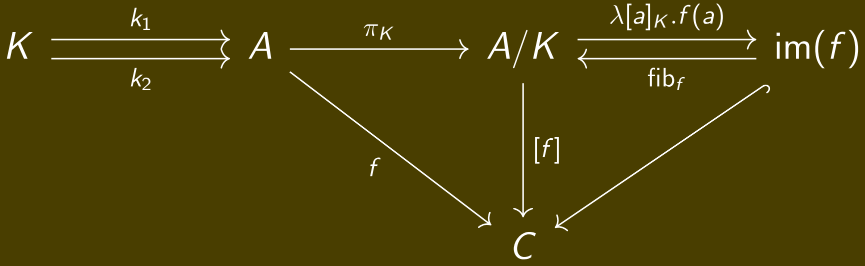
$$\mathrm{im}(f) = \{c \in C \mid c = f(a) \text{ for some } a \in A\} = \{c \in C \mid \mathrm{fib}_f(c) \neq \emptyset\}$$

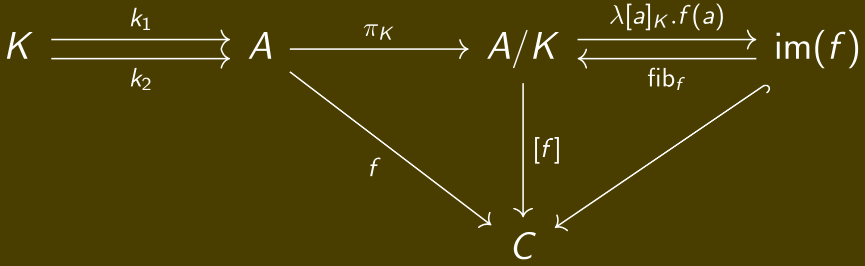
Prop. 6 For $f : A \to C$ with kernel K,

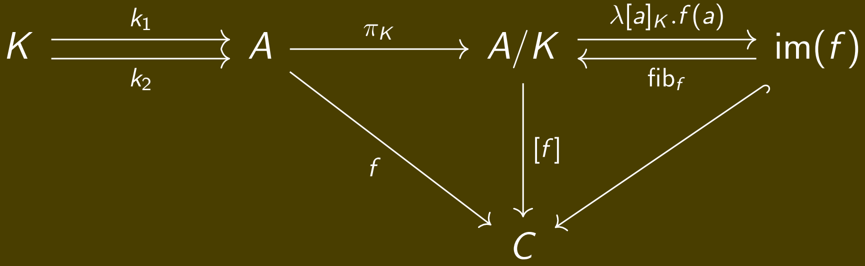
$$A/K = \{\mathrm{fib}_f(c) \mid c \in \mathrm{im}(f)\}$$

Prop. 7 $\mathrm{fib}_f : \mathrm{im}(f) \to A/K$ is a bijection:

$$A/K \underset{\mathrm{fib}_f}{\overset{\lambda[a]_K.f(a)}{\rightleftarrows}} \mathrm{im}(f)$$


K
A
A/K
im(f)
C
k1
k2
πK
λ[a]K.f(a)
fibf
f
[f]


K
k_1
k_2
A
π_K
A/K
$\lambda[a]_K.f(a)$
fib_f
$\mathrm{im}(f)$
f
$[f]$
C


K
k_1
k_2
A
π_K
A/K
λ[a]_K.f(a)
fib_f
im(f)
f
[f]
C

Check Your Understanding

- Prove Prop. 6
- Prove Prop. 7
- Prove that $[f] \circ \mathrm{fib}_f$ is equal to the inclusion of $\mathrm{im}(f)$ into C
- Prove that $[f]$ is injective

Summary

- Sets and functions between them

Summary

- Sets and functions between them
- The “algebra of functions”: composition and identity functions

Summary

- Sets and functions between them
- The "algebra of functions": composition and identity functions
- Special *functions* which satisfy universal mapping properties

Summary

- Sets and functions between them
- The "algebra of functions": composition and identity functions
- Special *functions* which satisfy universal mapping properties
- Special *sets* which satisfy universal mapping properties

Summary

- Sets and functions between them
- The "algebra of functions": composition and identity functions
- Special *functions* which satisfy universal mapping properties
- Special *sets* which satisfy universal mapping properties
- A notion of "element", defined in terms of functions

Summary

- Sets and functions between them
- The “algebra of functions”: composition and identity functions
- Special *functions* which satisfy universal mapping properties
- Special *sets* which satisfy universal mapping properties
- A notion of “element”, defined in terms of functions
- Constructions of set diagrams, with universal mapping properties

Summary

- Sets and functions between them
- The “algebra of functions”: composition and identity functions
- Special *functions* which satisfy universal mapping properties
- Special *sets* which satisfy universal mapping properties
- A notion of “element”, defined in terms of functions
- Constructions of set diagrams, with universal mapping properties
- A notion of “subset”, defined in terms of functions

Next Time. . .

Next Time. . .

- The Outer Limits of Set Theory

Next Time...

- The Outer Limits of Set Theory
- Categories

Next Time...

- The Outer Limits of Set Theory
- Categories
- Concrete Categories

Next Time. . .

- The Outer Limits of Set Theory
- Categories
- Concrete Categories
- Abstract Categories

Thanks for watching!