Categorial Logic in Lean

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Check out the website!

lean-catlogic.github.io

0 Completeness Proofs via LT Categories

Sketch of the idea

Proof Sketch To prove the soundness/completeness of a deductive calculus

1. Construct the Lindenbaum-Tarski category
   - Objects are (equiv. classes of) formulas, morphisms are 'lifted' from deductions
   - will have additional categorical structure corresponding to the logical structure of the theory
2. Give a natural bijection between models and and (structure-preserving) functors from LT-category into presheaf categories
3. Instantiate the bijection in 2 with the Yoneda embedding to get a *canonical model* which is *logically generic*

The Positive Propositional Calculus

top $\top$

$q \wedge p$

$p \to p$

implication

$p \wedge q$

q

p

$\top \to p$

$q \wedge q$

$p \wedge \top$

conjunction

atomic formula

The Positive Propositional Calculus

top $\top$

$q \land p$

$p \to p$

implication

$p \land q$

q

p

$\top \to p$

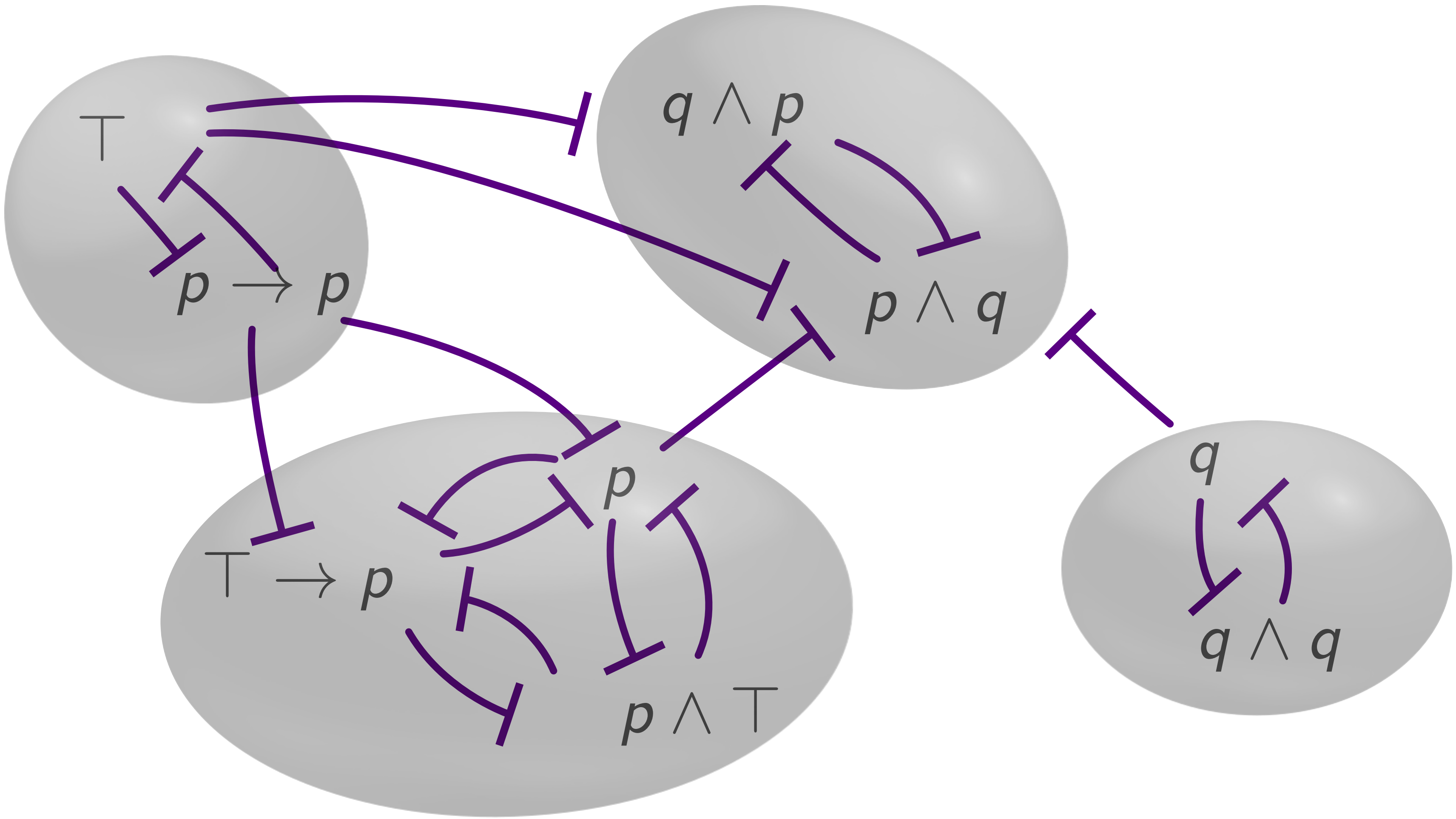
$p \land \top$

$q \land q$

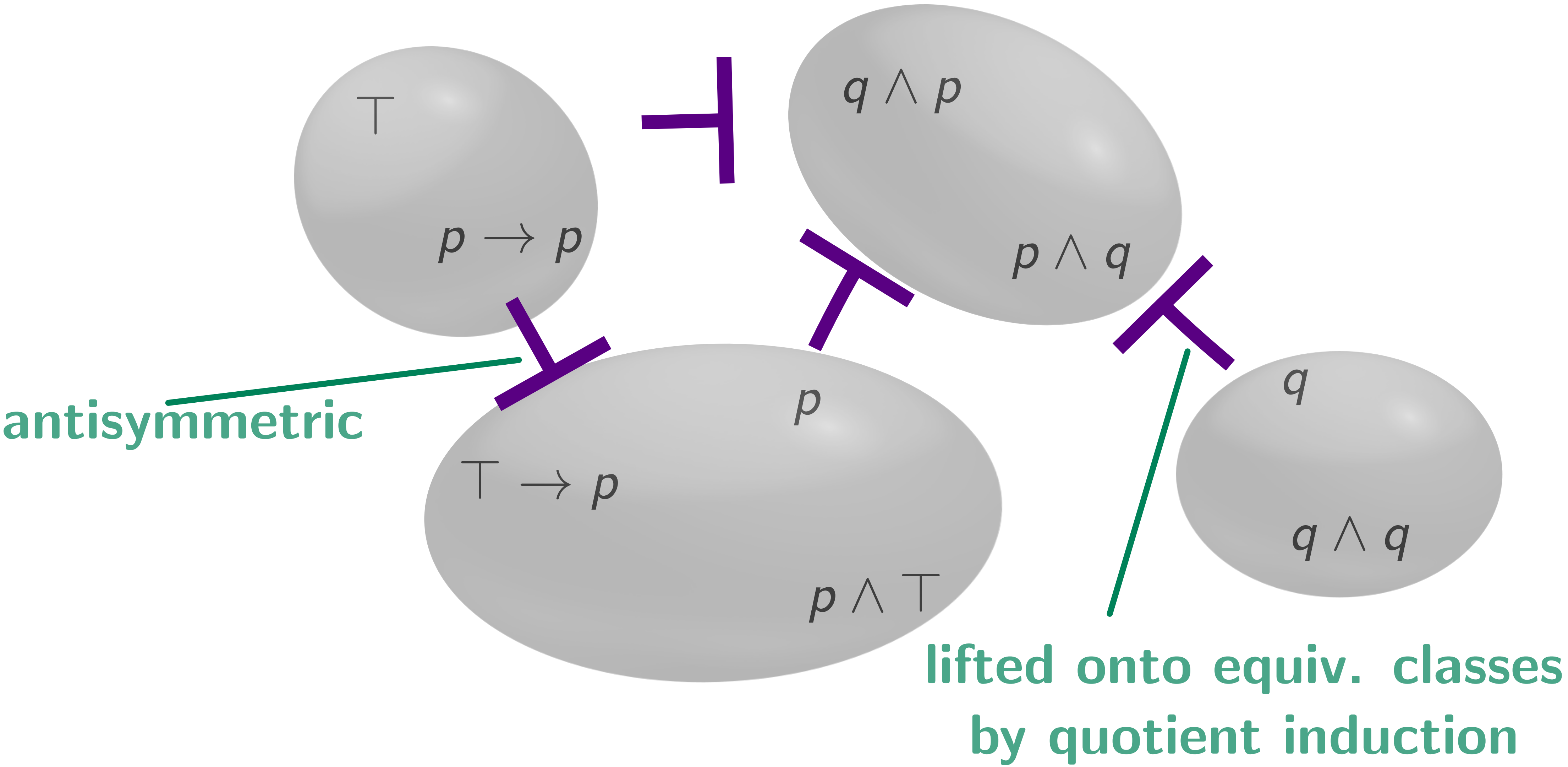
conjunction

atomic formula

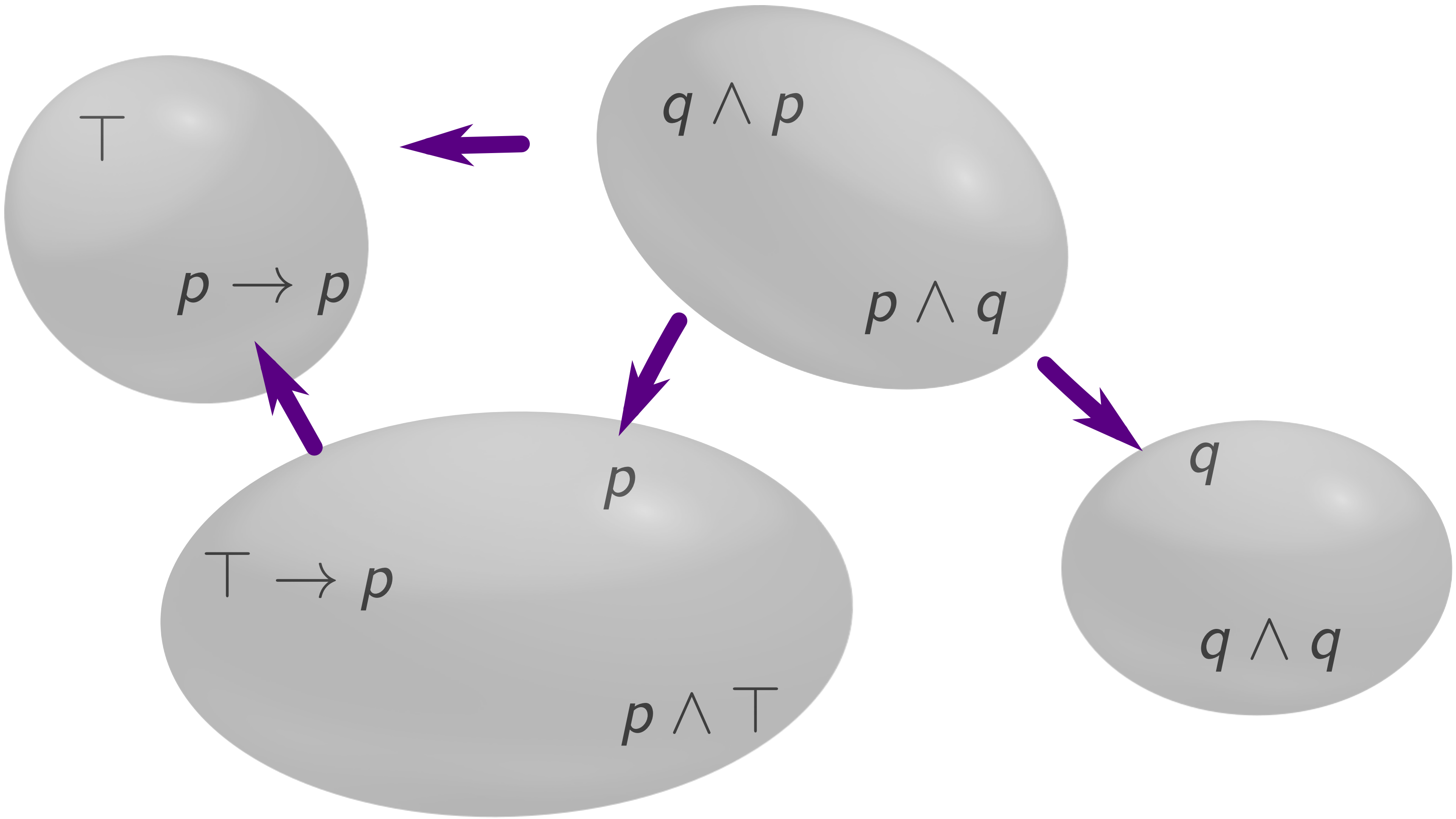
Quotient by inter-derivability and the syntactic category



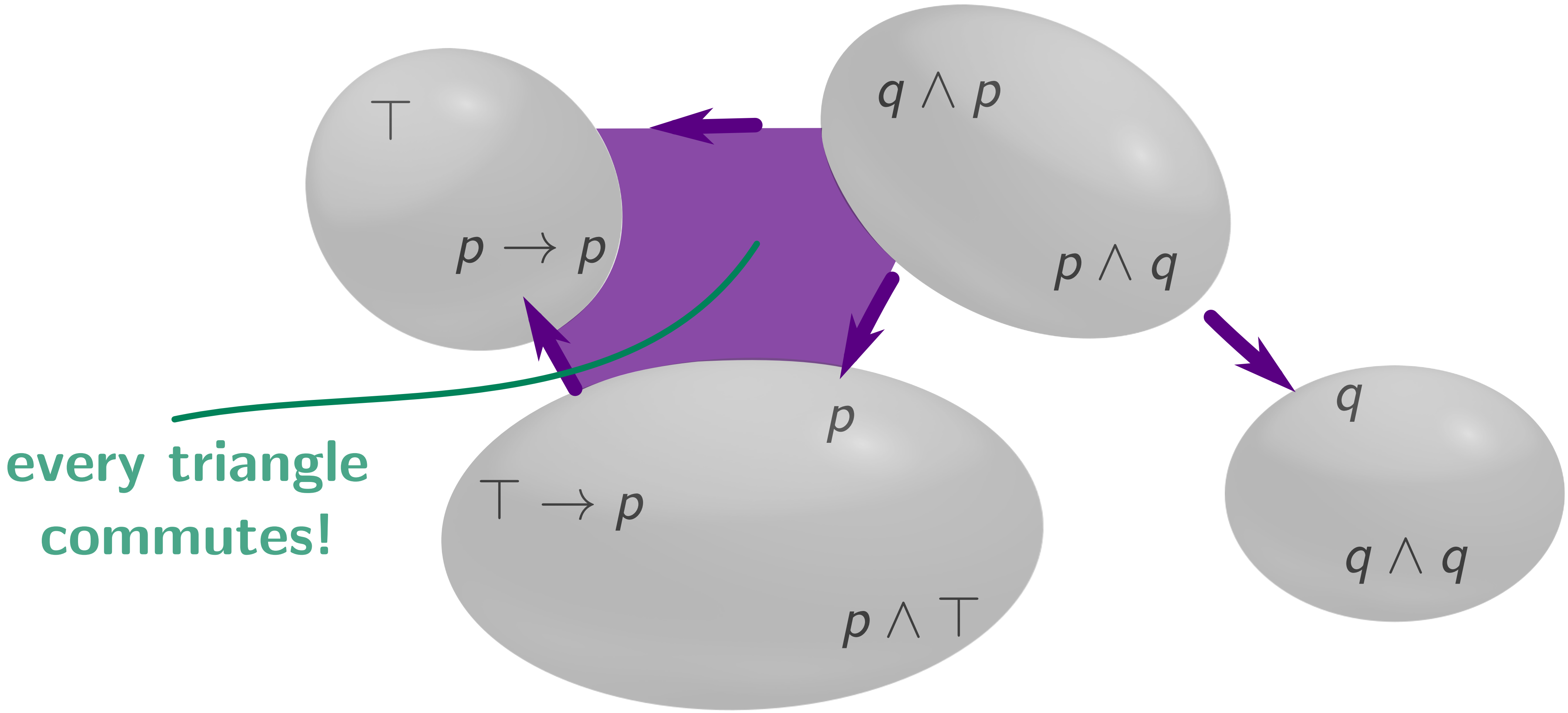
Quotient by inter-derivability and the syntactic category



Quotient by inter-derivability and the syntactic category



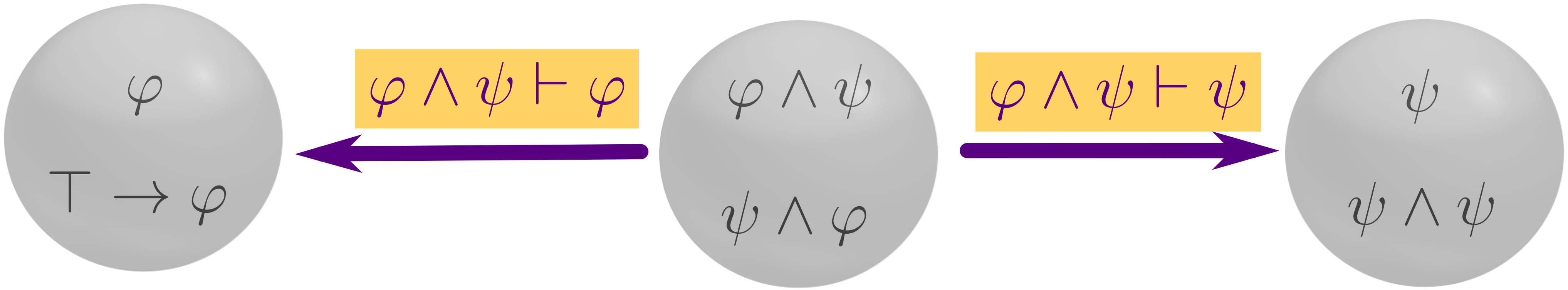
Quotient by inter-derivability and the syntactic category

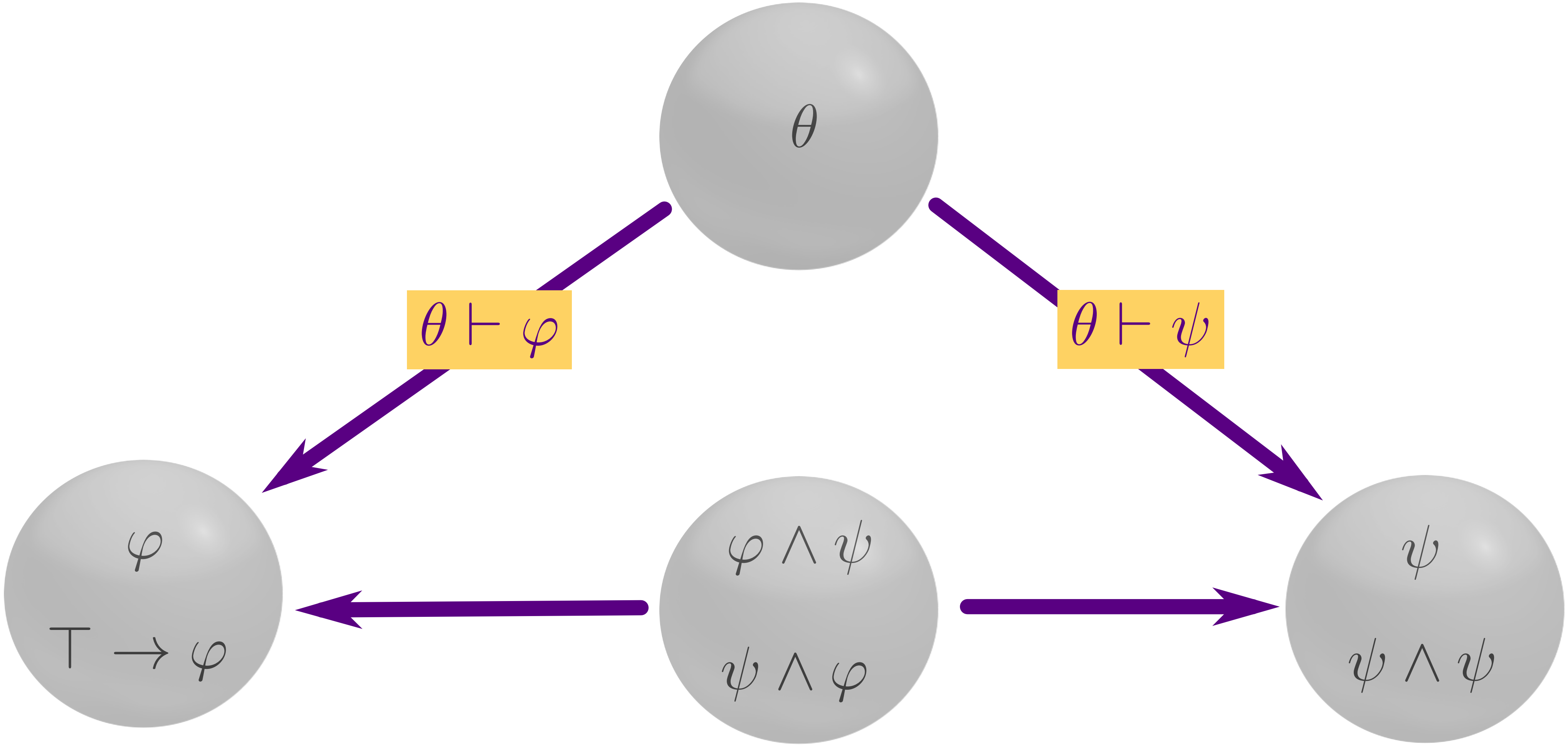


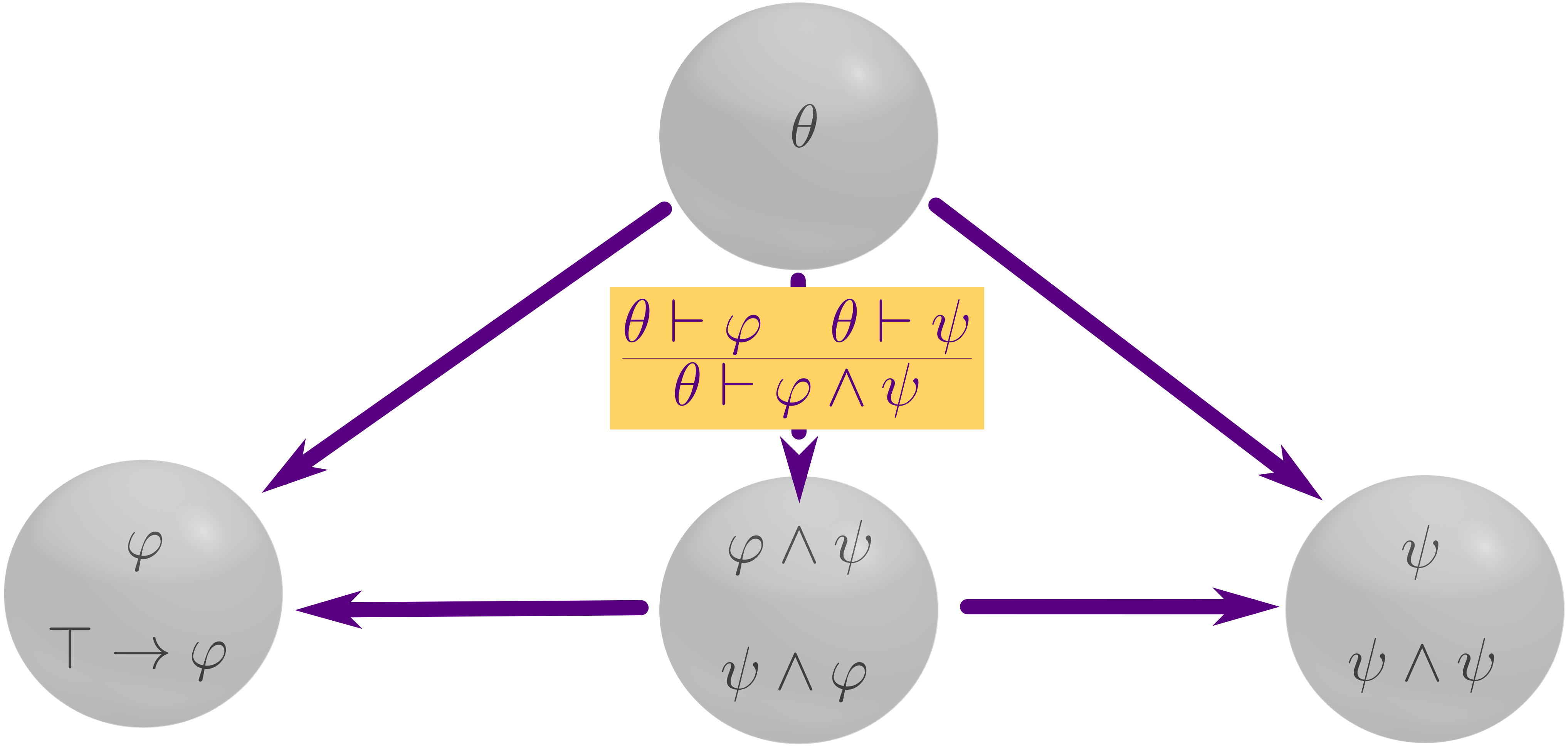
logical structure

lifts to

categorical structure


φ
$\top \to \varphi$
$\varphi \wedge \psi \vdash \varphi$
$\varphi \wedge \psi$
$\psi \wedge \varphi$
$\varphi \wedge \psi \vdash \psi$
ψ
$\psi \wedge \psi$


θ
$\theta \vdash \varphi$
$\theta \vdash \psi$
φ
$\top \to \varphi$
$\varphi \wedge \psi$
$\psi \wedge \varphi$
ψ
$\psi \wedge \psi$


θ
$\frac{\theta \vdash \varphi \quad \theta \vdash \psi}{\theta \vdash \varphi \wedge \psi}$
φ
$\top \to \varphi$
$\varphi \wedge \psi$
$\psi \wedge \varphi$
ψ
$\psi \wedge \psi$

Exercises:

- Verify that the equivalence class of $\top$ is the terminal object in the LT category
- Verify that these rules, when lifted onto the LT category, endow it with *exponentials*

$$\frac{\Phi \vdash \varphi \to \psi \quad \Phi \vdash \varphi}{\Phi \vdash \psi} \qquad \frac{\varphi \wedge \psi \vdash \theta}{\varphi \vdash \psi \to \theta}$$


Calculemus!

¡Formalicemos!

1 Formalization in Lean

The `lean-catLogic` project

- Content to formalize:
  - Awodey and Bauer's notes (awodey.github.io/catlog/notes), starting with Sect. 2.8
  - Awodey's Categorical Logic lectures, Proof and Computation Autumn School, Fischbachau, Germany, 2022 (awodey.github.io/fischbachau)
  - [MR95, HM92]
  - Eventually cover more of categorical logic...
  - Not aware of other formalizations of this material (?)
- Lean proof assistant [dMKA+15]
  - Lean 3 (current stable version), but may someday convert to Lean 4
  - Lean mathlib [mC20]
  - Tactic-based proofs, with utilities for custom tactics and metaprogramming in Lean itself

Agenda

- Original proof that LT category of the PPC is a CCC
  - Discuss shortcomings/limitations
- Optimizations (so far)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Representing the proof calculus I

- Formulas are defined as an inductive type `PPC_form`

1beaf90 languages/PPC.lean [permalink]

```
inductive PPC_form : Type
  | top : PPC_form
  | var : ℕ → PPC_form
  | and : PPC_form → PPC_form → PPC_form
  | impl : PPC_form → PPC_form → PPC_form
```

- The `derives` relation is an inductively-defined relation between sets of formulas and formulas

1beaf90 proof/PPC_natDeduct.lean [permalink]

Representing the proof calculus II

```
  | and_intro {Φ} {φ ψ : PPC_form}
      : derives Φ φ → derives Φ ψ → derives Φ (φ & ψ)
  | and_eliml {Φ} {φ ψ : PPC_form}
      : derives Φ (φ & ψ) → derives Φ φ
  | and_elimr {Φ} {φ ψ : PPC_form}
      : derives Φ (φ & ψ) → derives Φ ψ
```

- Define ⊢ relation between formulas as singleton-`derives`

1beaf90 proof/PPC_natDeduct.lean [permalink]

```
infix `⊢`:80 := λ φ ψ, derives {φ} ψ
```

Quotienting and forming the LT category

- The inter-derivability equivalence relation
- `PPC_eq := PPC_form / ⊣⊢`

1beaf90 semantics/PPC_poset.lean [permalink]

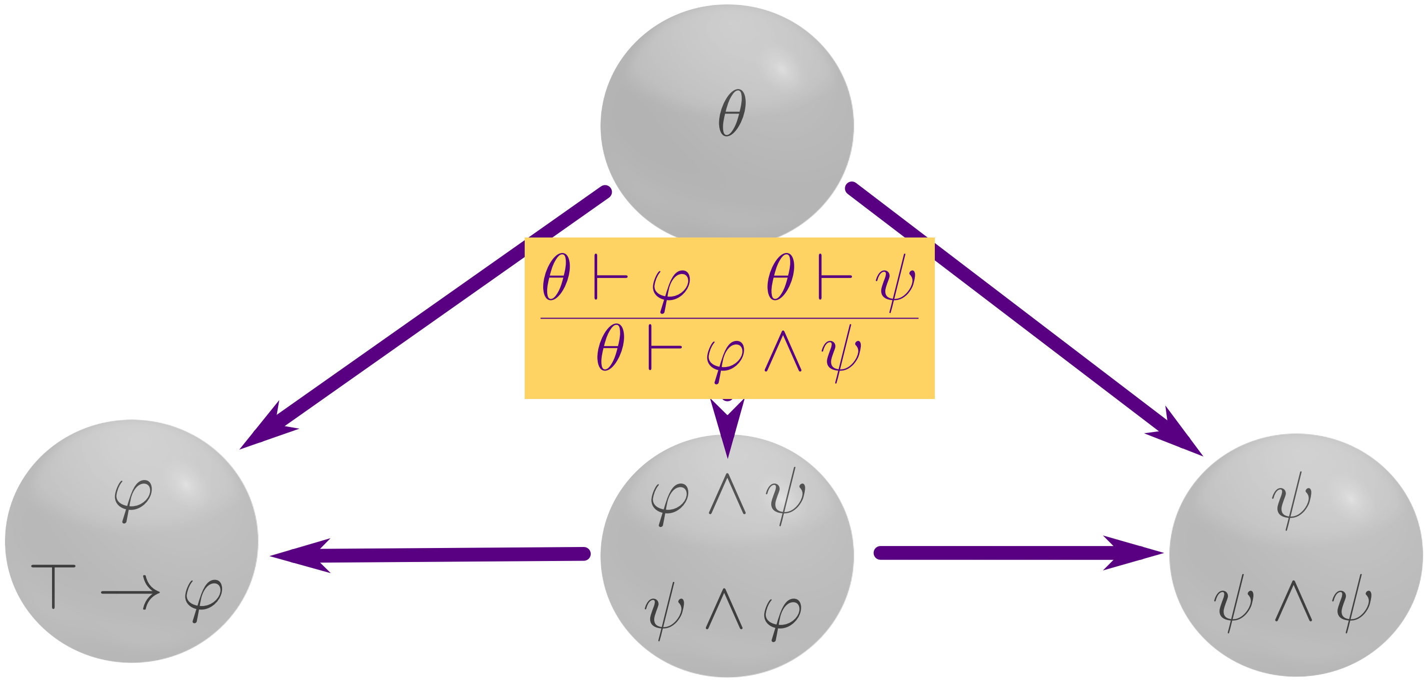
```
notation (name:=PPC_setoid.r) φ `⊣⊢` ψ :78 := PPC_setoid.r φ ψ

-- Quotienting PPC_form by inter-derivability
def PPC_eq : Type := quot (⊣⊢)
```

- Define the category:

1beaf90 semantics/PPC_syntacticCat.lean [permalink]

```
def ℂ_PPC : category (PPC_eq)
  := preorder.small_category PPC_eq
```

Proof it's a CCC



1beaf90 proof/PPC_natDeduct.lean [permalink]

```
  | and_intro {Φ} {φ ψ : PPC_form}
      : derives Φ φ → derives Φ ψ → derives Φ (φ & ψ)
```

Proof it's a CCC (cont.)

1beaf90 semantics/PPC_syntacticCat.lean [permalink]

```
  curry :=
    begin
      -- Actual construction
      assume X Y Z u,
      induction X with φ, induction Y with ψ, induction Z with θ,
      apply ℂ_PPC_hom,
      have h : (φ&ψ) ⊢ θ,
      exact (Exists.fst (ℂ_PPC_full u)),
      apply derives.impl_intro,
      apply and_Hyp_union,
      exact h,
      -- Proving this respects ⊣⊢
      apply funext, assume _, apply ℂ_PPC_thin,
      apply funext, assume _, apply ℂ_PPC_thin,
      apply funext, assume _, apply ℂ_PPC_thin,
```

Observation: Content is buried under all the boilerplate

Where's the actual content?

1beaf90 semantics/PPC_syntacticCat.lean [permalink]

```
pair :=
  begin
    -- Actual construction
    assume X Y Z f g,
    induction X with φ, induction Y with ψ, induction Z with χ,
    let h : χ ⊢ φ := le_of_hom f,
    let h' : χ ⊢ ψ := le_of_hom g,
    apply ℂ_PPC_hom,
    apply derives.and_intro,
    exact h, exact h',
    -- Proving this respects ⊣⊢
    apply funext, assume _, apply funext, assume _, apply ℂ_PPC_thin,
    apply funext, assume _, apply ℂ_PPC_thin,
    apply funext, assume _, apply ℂ_PPC_thin,
  end,
```

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Proof it's a CCC (cont.)

1beaf90 semantics/PPC_syntacticCat.lean [permalink]

```
  curry :=
    begin
      -- Actual construction
      assume X Y Z u,
      induction X with φ, induction Y with ψ, induction Z with θ,
      apply ℂ_PPC_hom,
      have h : (φ&ψ) ⊢ θ,
      exact (Exists.fst (ℂ_PPC_full u)),
      apply derives.impl_intro,
      apply and_Hyp_union,
      exact h,
      -- Proving this respects ⊣⊢
      apply funext, assume _, apply ℂ_PPC_thin,
      apply funext, assume _, apply ℂ_PPC_thin,
      apply funext, assume _, apply ℂ_PPC_thin,
```

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Let’s do better

2 The LiftT tactic

- Basic Lean proof: use tactics to solve goals, set new goals, etc.
- More advanced: use tactic combinators to automate repetitive aspects
- Even more advanced: define custom tactics using the `meta def` keyword
- Even more advanced: programming with the full power of the `tactic` monad

Recall: lifting boilerplate

1beaf90 semantics/PPC_syntacticCat.lean [permalink]

```
pair :=
  begin
    -- Actual construction
    assume X Y Z f g,
    induction X with φ, induction Y with ψ, induction Z with χ,
    let h : χ ⊢ φ := le_of_hom f,
    let h' : χ ⊢ ψ := le_of_hom g,
    apply ℂ_PPC_hom,
    apply derives.and_intro,
    exact h, exact h',
    -- Proving this respects ⊣⊢
    apply funext, assume _, apply funext, assume _, apply ℂ_PPC_thin,
    apply funext, assume _, apply ℂ_PPC_thin,
    apply funext, assume _, apply ℂ_PPC_thin,
  end,
```

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LiftT

$$\texttt{LiftT}: \texttt{tactic unit} \to \texttt{tactic unit}$$

```
apply derives.and_intro,
```

tactic performing the relevant action on deductions

LiftT: $\texttt{tactic unit} \to \texttt{tactic unit}$

```
apply derives.and_intro,
```

```
-- Actual construction
assume X Y Z f g,
induction X with φ, induction Y with ψ, induction Z with χ,
let h : χ ⊢ φ := le_of_hom f,
let h' : χ ⊢ ψ := le_of_hom g,
apply ℂ_PPC_hom,
apply derives.and_intro,
exact h, exact h',
-- Proving this respects ⊣⊢
apply funext, assume _, apply funext, assume _, apply ℂ_PPC_thin,
apply funext, assume _, apply ℂ_PPC_thin,
apply funext, assume _, apply ℂ_PPC_thin,
```

tactic constructing an operation on the LT category

The finished proof

bf4eda9 semantics/syntacticCat_cartesian.lean [permalink]

```
instance syn_FP_cat {Form : Type} [And : has_and Form] : FP_cat (Form _eq) :=
{
  unit := syn_obj And.top,
  term := by LiftT `[ apply And.truth ],
  unit_η := λ X f, by apply thin_cat.K,
  prod := and_eq,
  pr1 := by LiftT `[ apply And.and_eliml ],
  pr2 := by LiftT `[ apply And.and_elimr ],
  pair := by LiftT `[ apply And.and_intro ],
  prod_β1 := λ X Y Z f g, by apply thin_cat.K,
  prod_β2 := λ X Y Z f g, by apply thin_cat.K,
  prod_η :=  λ X Y, by apply thin_cat.K
}
```

The finished proof (cont.)

bf4eda9 semantics/syntacticCat_cartesian.lean [permalink]

```
instance syn_CC_cat {Form : Type} [Impl : has_impl Form] : CC_cat (Form _eq) :=
{
  exp := impl_eq,
  eval := by LiftT `[ apply cart_x.modus_ponens ],
  curry := by LiftT `[ apply cart_x.impl_ε],
  curry_β := λ {X Y Z} u, by apply thin_cat.K,
  curry_η := λ {X Y Z} v, by apply thin_cat.K,
}
```

What is LiftT doing?

lean-catlogic.github.io/docs/semantics/synCat_tactics

LiftT's agenda:

1. Introduce necessary variables, apply quotient induction, turn into a deduction problem
2. Apply input tactic
3. Clean up any auxiliary goals created in the previous step
4. Use the fact that the LT category is a poset to prove the coherences required by quotient induction

```
meta def LiftT
  (debugMode : parse (optional $ tk "?"))
  (debugPerformTac : parse (optional $ tk "!"))
  (debugCleanup : parse (optional $ tk "!"))
  (T : tactic unit)
  : tactic unit :=
do
  let proceedLevel :=
    match (debugMode,debugPerformTac,debugCleanup) with
    | (none,_,_) := 4 -- LiftT    invoked: do everything
    | (_,none,_) := 1 -- LiftT?   invoked: just do the assume's and syn_hom
    | (_,_,none) := 2 -- LiftT?!  invoked: also apply the tactic
    | _ := 3          -- LiftT?!! invoked: also do the first stage of cleanup
    end,
```

```
    -- Count & print how many objects and morphisms to assume
    (numobjs,nummor) ← doCount none,
    /- Assume objects and morphisms,
      - use induction to get that every object is of the form {|φ|} for some φ
  :Form
      - use syn_hom_inv to turn every assumed morphism into a derivation -/
    repeat_assume_induct (gen_nameList `φ_ numobjs),
    repeat_assume_replace `synCat.syn_hom_inv (gen_nameList `f_ nummor),
    -- Turn the synCat hom goal to a derivation goal
    applyc `synCat.syn_hom,
    trace_goal "MAIN GOAL",
  when (proceedLevel > 1) $ do
    pre_goal_count ← count_goals,
    -- Apply the input tactic
    T,
    -- Difference in goals
    post_goal_count ← count_goals,
```

```
    let relGoals : nat :=
      if pre_goal_count > post_goal_count
      then 0
      else (post_goal_count - pre_goal_count) + 1,
    trace_goals relGoals "POST-TACTIC GOALS",
  when (proceedLevel > 2) $ do
    -- Eliminate other goals (first stage of cleanup)
    iterate (
      (applyc `deduction_basic.derive_refl)
      <|> assumption
    ),
    trace_all_goals "CLEANUP GOALS",
  when (proceedLevel > 3) $ do
    -- Prove the coherences
    thin_cat.by_thin
```

Future directions

- Formalize the rest of the Kripke Completeness proof
- Adjacent topics
  - Stone Duality
  - IPL, Beth models, and Heyting Algebras
  - Topological semantics
  - Lambda Calculus
  - Propositional modal logic
- Main topics of categorical logic
  - Lawvere Algebraic Theories
  - Predicate Logic
- Improve LiftT's operation
- Use mathlib in more places

[dMKA+15] Leonardo de Moura, Soonho Kong, Jeremy Avigad, Floris Van Doorn, and Jakob von Raumer.
The lean theorem prover (system description).
In *Automated Deduction-CADE-25: 25th International Conference on Automated Deduction, Berlin, Germany, August 1-7, 2015, Proceedings 25*, pages 378–388. Springer, 2015.

[HM92] Victor Harnik and Michael Makkai.
Lambek's categorical proof theory and läuchli's abstract realizability.
The Journal of symbolic logic, 57(1):200–230, 1992.

[mC20] The mathlib Community.
The lean mathematical library.
In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, CPP 2020, page 367–381, New York, NY, USA, 2020. Association for Computing Machinery.

[MR95] Michael Makkai and Gonzalo E. Reyes.
Completeness results for intuitionistic and modal logic in a categorical setting.
Annals of Pure and Applied Logic, 72(1):25–101, 1995.

Thank You!!

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Diamond modality

bf4eda9 deduction/deduction_monadic.lean [permalink]

```
class has_diamond (Form : Type) extends has_struct_derives Form :=
  (diamond : Form → Form)
  (dmap  : ∀ {Φ : Hyp}{φ ψ : Form},
    derives (insert φ Φ) ψ → derives (insert (diamond φ) Φ) (diamond ψ))
  (dpure : ∀ {Φ : Hyp} {φ : Form},
    derives Φ φ → derives Φ (diamond φ))
  (djoin : ∀ {Φ : Hyp} {φ : Form},
    derives Φ (diamond (diamond φ)) → derives Φ (diamond φ))

notation (name:= has_diamond.diamond) `◇`:81 φ := has_diamond.diamond φ
```

Gives rise to a monad!

bf4eda9 semantics/syntacticCat_monadic.lean [permalink]

```
def diamond_monad {Form : Type} [Diam : has_diamond Form] :
  category_theory.monad (Form _eq) :=
{
  obj := diamond_eq,
  map := by LiftT `[ apply Diam.dmap ],
  η'  := ⟨
          by LiftT `[ apply Diam.dpure ],
          λ X Y f, by apply thin_cat.K,
         ⟩,
  μ' :=  ⟨
          by LiftT `[ apply Diam.djoin ],
          λ X Y f, by apply thin_cat.K,
         ⟩,
}
```