Updates on Paranatural Category Theory

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Premise: A *category theory* of strong dinaturality

Defn. A **difunctor** on a category $\mathbb{C}$ is a functor $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$.

Defn. Given difunctors Γ, Δ, a **strong dinatural transformation** α from Γ to Δ is a family of maps

$$\alpha_I \colon \Gamma(I, I) \to \Delta(I, I)$$

for each object I of $\mathbb{C}$, such that, for every $f : \mathbb{C}(I, J)$, $h \colon \Gamma(I, I)$, $k \colon \Gamma(J, J)$,

$$\Gamma(I, f)\ h = \Gamma(f, J)\ k \qquad \text{implies} \qquad \Delta(I, f)\ (\alpha_I\ h) = \Delta(f, J)\ (\alpha_J\ k)$$

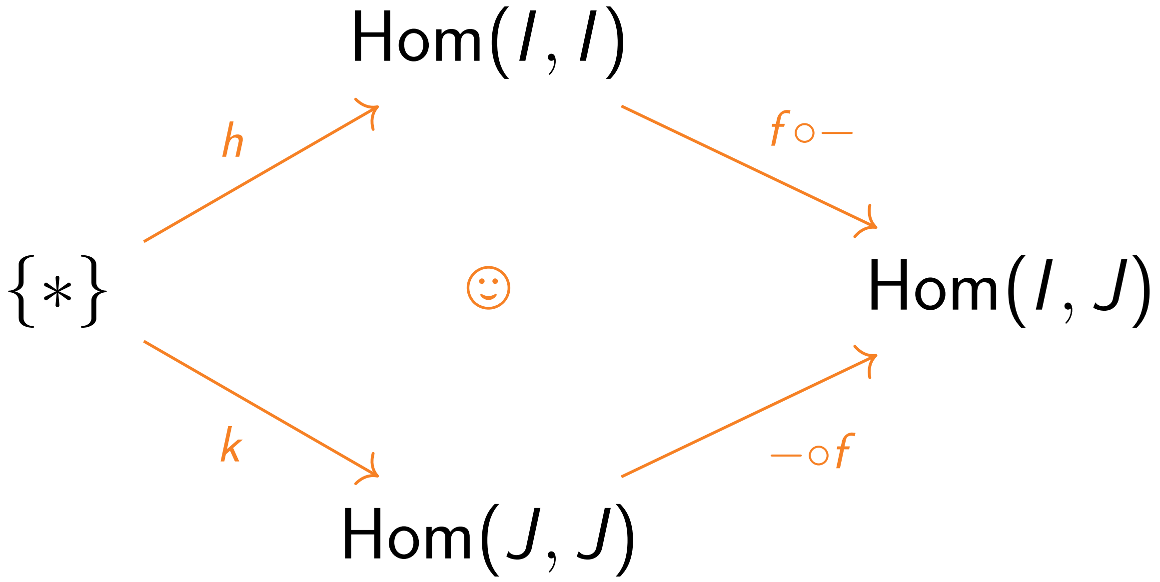
Fact The identity maps form a strong dinatural transformation.

Fact Strong dinatural transformations are closed under (pointwise) composition.

Main example: Church numerals

$$(\overline{n})_I : \mathrm{Hom}(I, I) \to \mathrm{Hom}(I, I)$$
$$(\overline{n})_I = \lambda h \to h^n$$

$$f \circ h = k \circ f$$



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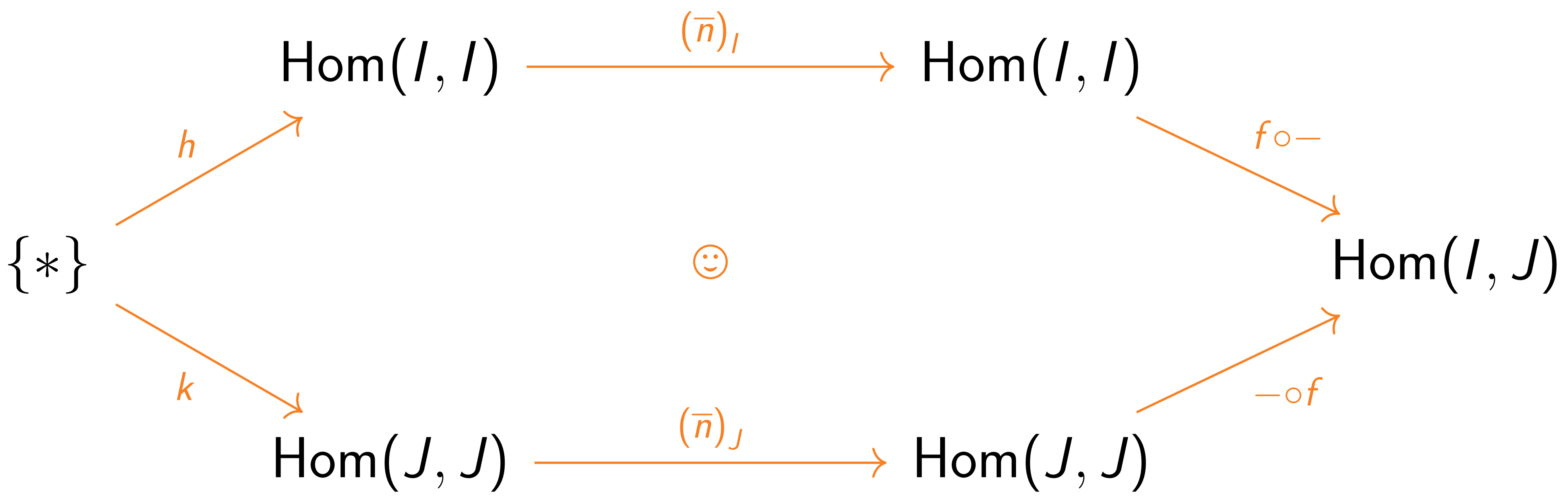

Basic definitions

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Fact The identity maps form a strong dinatural transformation.

Fact Strong dinatural transformations are closed under (pointwise) composition.

Notation Write

$$\Gamma \overset{\diamond}{\to} \Delta \qquad \text{or} \qquad \int_{I\,:\,\mathbb{C}} \Gamma(I, I)\, \mathbf{d}\Delta(I, I)$$

for the set of strong dinatural transformations from Γ to Δ.

while the object e itself, by abuse of language, is called the "end" of S and is written with integral notation as

$$e = \int_c S(c, c) = \text{End of } S\,.$$

Note that the "variable of integration" c appears twice under the integral sign (once contravariant, once covariant) and is "bound" by the integral sign, in that the result no longer depends on c and so is unchanged if "c" is replaced by any other letter standing for an object of the category C. These properties are like those of the letter x in the usual integral $\int f(x)\,dx$ of the calculus.

[ML78, Chapter IX]

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$$\int_{I:\mathbb{C}} \Gamma(I, I)\ \mathbf{d}\Delta(I, I) = \sum_{\text{(diagonal family)}} \prod_{\text{(structural morphism)}} \cdots$$

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∫trong dinatural

Why we (might) care:

Category theory for parametricity

Local professor discovers this *one weird trick* to get theorems FOR FREE!

THEOREM SALESPEOPLE *HATE HIM*

Find out how: [Wad89]

- Wadler applied Reynolds's parametricity result [Rey83] to obtain "free theorems" — theorems that hold for *all* values of a type, regardless of how they're implemented
- Interesting: parametricity is stated in terms of *relations*, but used by instantiating those relations to *functions*

- $t\colon \forall X.\text{List } X \to \text{List } X$

$$(\text{map } f) \circ t_I = t_J \circ (\text{map } f)$$

  for all $f\colon I \to J$
- $e\colon \forall X.(X \to \text{Bool}) \to (\text{List } X \to \text{Bool})$

$$(e_I\ (q \circ f)) = (e_J\ q) \circ (\text{map } f)$$

  for all $f\colon I \to J$, $q\colon J \to \text{Bool}$

Is parametricity just naturality?

No: This doesn't work with mixed variance

Diagonal naturality?

- Consider $\forall X.(X \to X) \to (X \to X)$. $\mathrm{Hom}\colon \mathrm{Set}^{\mathrm{op}} \times \mathrm{Set} \to \mathrm{Set}$, so a natural transformation $\alpha\colon \mathrm{Hom} \to \mathrm{Hom}$ would be *double* indexed over objects of Set:

$$\alpha_{(I,J)}\colon \mathrm{Hom}(I,J) \to \mathrm{Hom}(I,J)$$

- Dinatural transformations [DS70],[ML78, Chapter IX] have the right shape:

$$\alpha_I\colon \mathrm{Hom}(I,I) \to \mathrm{Hom}(I,I)$$

but. . .

  - Their "naturality" condition is super weird: for all $f\colon I \to J$

$$\text{for all } f'\colon J \to I, \qquad f \circ (\alpha_I(f' \circ f)) = \alpha_J(f \circ f') \circ f$$

  - Dinaturals don't compose

Free Theorem For any $t\colon \forall X.(X \to X) \to (X \to X)$, any $f\colon I \to J$, $h\colon I \to I$, $k\colon J \to J$,

$$f \circ h = k \circ f \qquad \text{implies} \qquad f \circ (t_I\ h) = (t_J\ k) \circ f$$

Free Theorem For any $s\colon \forall X.(X \times X \to \mathrm{Bool}) \to (\mathrm{List}\ X \to \mathrm{List}\ X)$,any $f\colon I \to J$, $\prec_I\colon I \times I \to \mathrm{Bool}$, $\prec_J\colon J \times J \to \mathrm{Bool}$, $xs\colon \mathrm{List}\ I$

$$(\prec_J) \circ (f \times f) = (\prec_I) \qquad \text{implies} \qquad s_J\ (\prec_J)\ (\mathrm{map}\ f\ xs) = \mathrm{map}\ f\ (s_I\ (\prec_I)\ xs)$$

Main example: Church numerals

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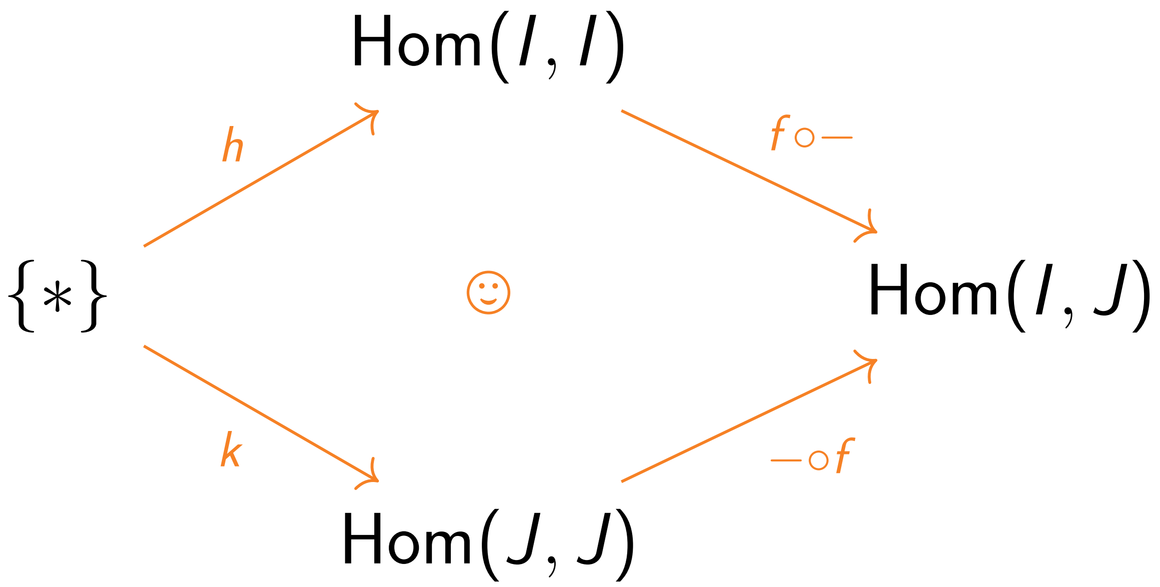

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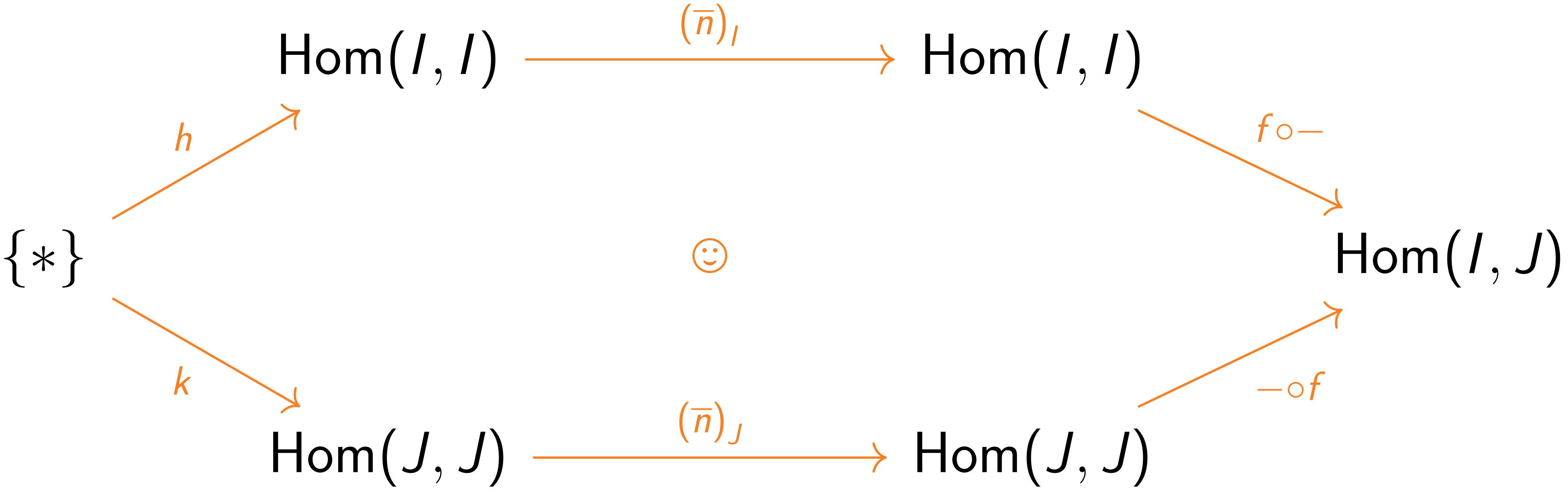

Strong dinaturality returns

Free Theorem For any $t \colon \forall X.(X \to X) \to (X \to X)$, any $f \colon I \to J$, $h \colon I \to I$, $k \colon J \to J$,

$$f \circ h = k \circ f \qquad \text{implies} \qquad f \circ (t_I\ h) = (t_J\ k) \circ f$$

Free Theorem For any $s \colon \forall X.(X \times X \to \text{Bool}) \to (\text{List}\ X \to \text{List}\ X)$,any $f \colon I \to J$, $\prec_I \colon I \times I \to \text{Bool}$, $\prec_J \colon J \times J \to \text{Bool}$, $xs \colon \text{List}\ I$

$$(\prec_J) \circ (f \times f) = (\prec_I) \qquad \text{implies} \qquad s_J\ (\prec_J)\ (\text{map}\ f\ xs) = \text{map}\ f\ (s_I\ (\prec_I)\ xs)$$

So, parametricity is strong dinaturality, right?

Divergence between strong dinaturality and parametricity

Consider

$$\phi\colon \forall X.((X \to X) \to X) \to X$$

Free Theorem For all $f : I \to J$, $p\colon (I \to I) \to I$, $q\colon (J \to J) \to J$,

$$\left[\forall h\ k, f \circ h = k \circ f \quad \text{implies} \quad f(p\ h) = q\ k\right] \quad \text{implies} \quad f(\phi_I\ p) = \phi_J\ q$$

ϕ is a strong dinatural transformation $\int_X ((X \to X) \to X)\,\mathbf{d}X$ if, for all f, p, q,

$$\left[\forall r\colon J \to I, f(p(r \circ f)) = q(f \circ r)\right] \quad \text{implies} \quad f(\phi_I\ p) = \phi_J\ q$$

What to do?

1. Give up!
2. Rule out types like $\forall X.((X \to X) \to X) \to X$.

> types entails strong dinaturality [9]. For the purposes of this paper, we assume that all recursion operators of interest are strongly dinatural; in practice, we are not aware of any such operators in common use where this assumption fails.

[HH15]

3. Give difunctors a true exponential

Consider

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Free Theorem For all $f : I \to J$, $p\colon (I \to I) \to I$, $q\colon (J \to J) \to J$,

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Idea: Copy from the theory of presheaves

Define the diYoneda embedding $\mathbf{yy}\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$,

$$\mathbf{yy}\,(I, J)\,(K, L) = \mathbb{C}(I, L) \times \mathbb{C}(K, J)$$

Lemma For $F\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$,

$$F(I, J) \cong \int_K \mathbb{C}(J, K) \times \mathbb{C}(K, I)\, \mathbf{d}F(K, K)$$

strong dinatural in I, J.

Given difunctors S, T, a "Yoneda calculation" tells us what the exponential S^T *should be*:

$$\begin{aligned} S^T\,(I, J) &\cong \int_K \mathbb{C}(J, K) \times \mathbb{C}(K, I)\, \mathbf{d}S^T(K, K) \\ &\cong \int_K \mathbb{C}(J, K) \times \mathbb{C}(K, I) \times T(K, K)\, \mathbf{d}S(K, K) \end{aligned}$$

Problem: The diYoneda Lemma is false!

Trying to prove it

Lemma For $F \colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$,

$$F(I, I) \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} \int_K \mathbb{C}(I, K) \times \mathbb{C}(K, I) \, \mathbf{d}F(K, K)$$

strong dinatural in I, J.

- ✓ $x \mapsto \lambda K \; (a, b) \to F(b, a) \; x$
- ✓ $\phi \mapsto \phi_I(\mathrm{id}, \mathrm{id})$
- ✓ $x = (\lambda K \; (a, b) \to F(b, a) \; x)_I \; (\mathrm{id}, \mathrm{id})$
- X $\phi = \lambda K \; (a, b) \to F(b, a) \; (\phi_I(\mathrm{id}, \mathrm{id}))$

Counterexample

$$(\lambda K \; (a, b) \to (b \circ a)^2) \quad : \quad \int_K \mathsf{Set}(I, K) \times \mathsf{Set}(K, I) \, \mathbf{d}\mathrm{Hom}_{\mathsf{Set}}(K, K)$$

Moving forward

- Strong dinatural transforms $\mathbf{yy}(I, I) \xrightarrow{\diamond} F$ contain more info than just $F(I, I)$.
  Conj?

$$\mathrm{HomSet} \times \mathbb{N} \cong \int_K \mathrm{Set}(I, K) \times \mathrm{Set}(K, I)\, \mathbf{d}\mathrm{Hom}_{\mathrm{Set}}(K, K)$$

- Lots of surrounding theory to build up
  - Connection to initial algebras: [Uus10, AFS18]
  - Dual: strong coends, existential types, terminal coalgebras
  - Strong (co)end calculus, à la [Lor23]

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Thank you!