Paranatural Category Theory

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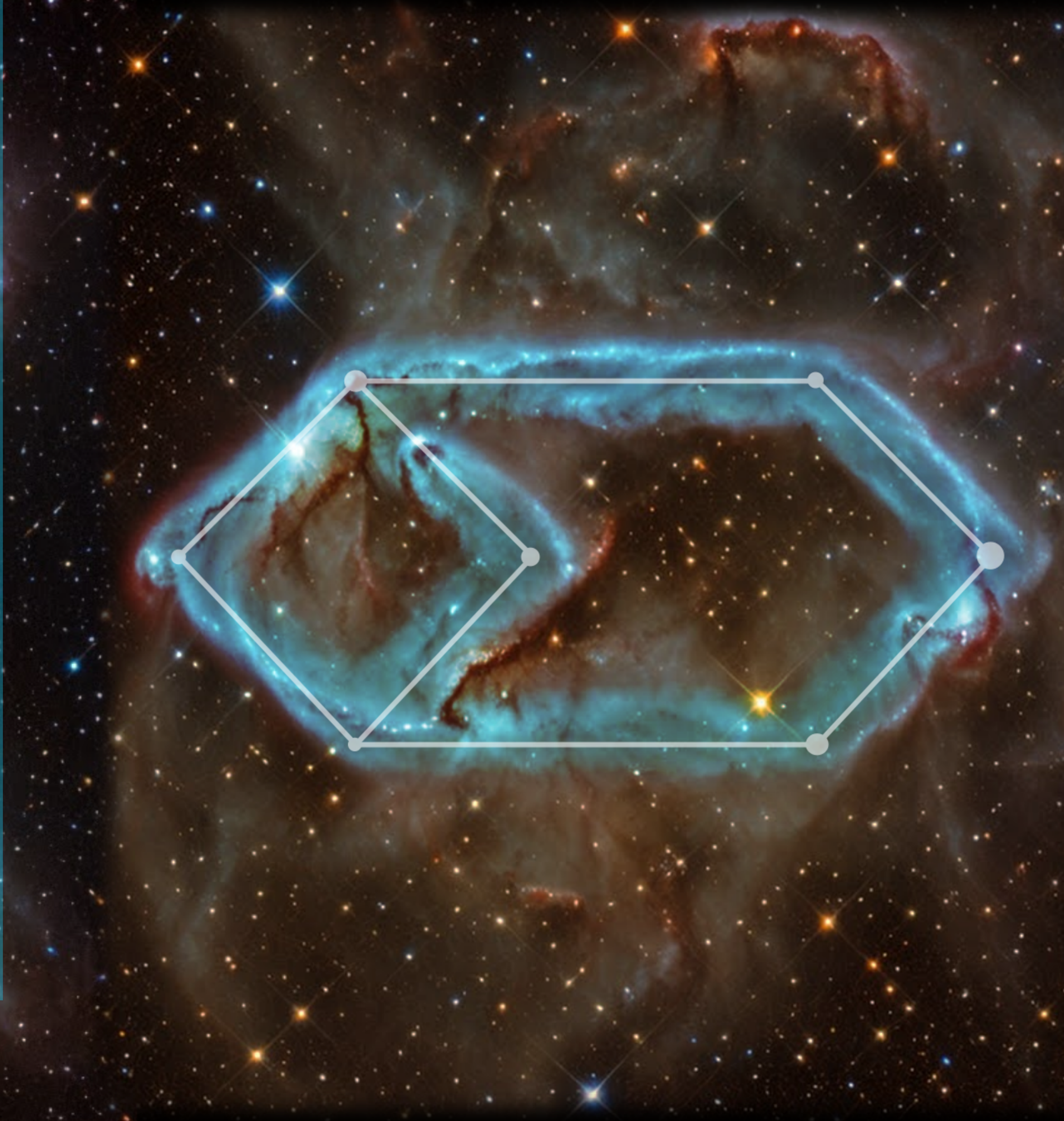

Question: What is the appropriate notion of *transformation* between difunctors?

Church numerals as endo-transforms of Hom

Consider the difunctor $\mathrm{Hom}\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$. Then, for each $n \in \mathbb{N}$, we can define the family of maps

$$\lambda f.f^n \quad : \mathrm{Hom}(J, J) \to \mathrm{Hom}(J, J)$$

indexed over objects J of $\mathbb{C}$.

I'll use the term **difunctor** to refer to functors of the form $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$

Notion of "transformation" between difunctors Γ, Δ:

- **natural transformation**: for every $I, J \in \mathbb{C}$, a function $\alpha_{I,J} \colon \Gamma(I, J) \to \Delta(I, J)$, satisfying naturality
  - Doesn't capture 'diagonal' transformations, e.g. the Church numerals
- **dinatural transformation**: for every $J \in \mathbb{C}$, a function $\alpha_J \colon \Gamma(J, J) \to \Delta(J, J)$, satisfying a "dinaturality condition".
  - Too weak—dinaturals don't compose (in general)
- **paranatural transformation**: for every $J \in \mathbb{C}$, a function $\alpha_J \colon \Gamma(J, J) \to \Delta(J, J)$, satisfying a "paranaturality condition".

Dinatural Transformations

For every $i_2 \in \mathrm{Hom}(I_0, I_1)$

$$\begin{tikzcd}
 & \Gamma(I_0, I_0) \arrow[r, "\alpha_{I_0}"] & \Delta(I_0, I_0) \arrow[dr, "\Delta(I_0, i_2)"] & \\
\Gamma(I_1, I_0) \arrow[ur, "\Gamma(i_2, I_0)"] \arrow[dr, "\Gamma(I_1, i_2)"'] & & & \Delta(I_0, I_1) \\
 & \Gamma(I_1, I_1) \arrow[r, "\alpha_{I_1}"'] & \Delta(I_1, I_1) \arrow[ur, "\Delta(i_2, I_1)"'] &
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 & \Gamma(l_0,l_0) \arrow[rr, "\alpha_{l_0}"] & & \Delta(l_0,l_0) \arrow[dr, "\Delta(l_0,i_2)"] \arrow[rr, "\beta_{l_0}"] & & \Theta(l_0,l_0) \arrow[dr, "\Theta(l_0,i_2)"] & \\
\Gamma(l_1,l_0) \arrow[ur, "\Gamma(i_2,l_0)"] \arrow[dr, "\Gamma(l_1,i_2)"'] & & \Delta(l_1,l_0) \arrow[ur, "\Delta(i_2,l_0)"] \arrow[dr, "\Delta(l_1,i_2)"'] & & \Delta(l_0,l_1) & & \Theta(l_0,l_1) \\
 & \Gamma(l_1,l_1) \arrow[rr, "\alpha_{l_1}"'] & & \Delta(l_1,l_1) \arrow[ur, "\Delta(i_2,l_1)"'] \arrow[rr, "\beta_{l_1}"'] & & \Theta(l_1,l_1) \arrow[ur, "\Theta(i_2,l_1)"'] &
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Defn. Given difunctors Γ, Δ, we say a family of maps $\alpha_J \colon \Gamma(J, J) \to \Delta(J, J)$ is a **paranatural transformation** from Γ to Δ and write $\alpha \colon \Gamma \xrightarrow{\diamond} \Delta$ if, for every $i_2 \in \mathrm{Hom}(I_0, I_1)$, the following hexagon commutes.

$$
\begin{tikzcd}
 & \Gamma(I_0, I_0) \arrow[rr, "\alpha_{I_0}"] \arrow[dr, "\Gamma(I_0, i_2)"] & & \Delta(I_0, I_0) \arrow[dr, "\Delta(I_0, i_2)"] & \\
P \arrow[ur, "\pi_0"] \arrow[dr, "\pi_1"'] \arrow[r, phantom, ">"] & & \Gamma(I_0, I_1) & & \Delta(I_0, I_1) \\
 & \Gamma(I_1, I_1) \arrow[rr, "\alpha_{I_1}"'] \arrow[ur, "\Gamma(i_2, I_1)"'] & & \Delta(I_1, I_1) \arrow[ur, "\Delta(i_2, I_1)"'] &
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1 \arrow[ur, "\gamma_0"] \arrow[dr, "\gamma_1"'] & & \Gamma(I_0, I_1) & & \Delta(I_0, I_1) \\
 & \Gamma(I_1, I_1) \arrow[rr, "\alpha_{I_1}"'] \arrow[ur, "\Gamma(i_2,I_1)"'] & & \Delta(I_1, I_1) \arrow[ur, "\Delta(i_2,I_1)"'] &
\end{tikzcd}
$$

"if the diamond commutes, so does the hexagon"

Some Results

Prop. The Church numerals are paranatural transformations $\mathrm{Hom} \xrightarrow{\diamond} \mathrm{Hom}$

Prop. If $\alpha \colon \Gamma \xrightarrow{\diamond} \Delta$ and $\beta \colon \Delta \xrightarrow{\diamond} \Theta$, then the pointwise-defined composite $(\beta \circ \alpha)_I := \beta_I \circ \alpha_I$ is a paranatural transformation $\Gamma \xrightarrow{\diamond} \Theta$

Defn. Write $\overset{\diamond}{\mathbb{C}}$ for the category whose objects are difunctors and whose morphisms are paranatural transformations.

Prop. $\overset{\diamond}{\mathbb{C}}$ has all finite products

Prop. $\overset{\diamond}{\mathbb{C}}$ is cartesian closed

Conj. $\overset{\diamond}{\mathbb{C}}$ is an elementary topos

Conj. $\mathrm{Hom}^{\mathrm{Hom}}$ is a natural numbers object in $\overset{\diamond}{\mathbb{C}}$

Defn. The **diYoneda embedding yy** : $\mathbb{C}^{\text{op}} \times \mathbb{C} \to \overset{\diamond}{\mathbb{C}}$ is the functor whose object part is given by

$$\mathbf{yy}\,(I_0, I_1)\,(J_0, J_1) := \mathrm{Hom}(I_0, J_1) \times \mathrm{Hom}(J_0, I_1)$$

and whose four morphism parts are given by appropriate pre- and post-compositions.

Lemma For any difunctor $\Delta\colon \mathbb{C}^{\text{op}} \times \mathbb{C} \to \mathsf{Set}$, there is a bijection

$$\Delta(I, J) \quad \cong \quad \mathbf{yy}(J, I) \overset{\diamond}{\to} \Delta$$

paranatural in I, J.

- Note that I and J are flipped on the right
- To prove this, we construct an $\alpha_d\colon \mathbf{yy}(I, I) \overset{\diamond}{\to} \Delta$ for each $d\colon \Delta(I, I)$ and vice-versa.

Claim The category of difunctors $\overset{\diamond}{\mathbb{C}}$ has exponential objects.

Proof. By "diYoneda reasoning": for difunctors Γ, Δ, *suppose* their exponential Δ^Γ existed. Then

$$\begin{aligned} \Delta^\Gamma(I, J) &\cong \mathbf{yy}(J, I) \overset{\diamond}{\to} \Delta^\Gamma && \text{diYoneda Lemma} \\ &\cong \mathbf{yy}(J, I) \times \Gamma \overset{\diamond}{\to} \Delta && \text{(desired property)} \end{aligned}$$

so now *define* $\Delta^\Gamma(I, J)$ to be $\mathbf{yy}(J, I) \times \Gamma \overset{\diamond}{\to} \Delta$, and verify this satisfies all the necessary properties.

Have we actually done anything new here?

Is paranaturality an instance of naturality?

Question Given difunctors $\Gamma, \Delta \colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$, can we define functors

$$\overline{\Gamma}, \overline{\Delta} \colon \mathbb{C}' \to \mathbb{D}$$

(for some appropriately-picked $\mathbb{C}', \mathbb{D}$) such that *paranatural transformations* $\Gamma \xrightarrow{\diamond} \Delta$ are the same thing as *natural transformations* $\overline{\Gamma} \to \overline{\Delta}$?

- Positive: No need to develop paranatural category theory separately, diYoneda is just an instance of Yoneda, $\overset{\diamond}{\mathbb{C}}$ “is” a presheaf category
- Negative: Paranatural category theory is indeed a novel branch of category theory, diYoneda is a distinct result from Yoneda, difunctor categories may be differently-behaved than presheaf categories

Further topics in paranatural category theory

- Equivalent formulations of paranaturality
- Categories of diagonal elements ("Γ-structures")
- "Splice categories"
- Strong (Co)End calculus
  - Structural ends
  - Initial algebras and [Uus10]'s Yoneda-like lemma
  - Structural coends, terminal coalgebras, and bisimulations
- Dependent paranatural transformations (and maybe a dependent diYoneda Lemma?)

Applications

- Mathematical framework for categorical semantics of (co)inductive types (generalizing and dualizing [AFS18])
- Parametricity: Paranatural transformations encode impredicative 'universal' and 'existential' types (e.g. from System F)
  - Paranatural transformations correspond to parametrically polymorphic functions, with the paranaturality condition matching the 'free theorems' of [Wad89]
  - Structural coends encode 'abstract data structures'
- Difunctor models of type theory
  - Uses diYoneda for "lifting Grothendieck universes", à la [HS99]

Sources

- Collection of links: jacobneu.github.io/research/paranat
- arXiv preprint: arxiv.org/abs/2307.09289
- HoTTEST talk:
  - Video: youtube.com/watch?v=X4v5HnnF2-o
  - Slides: research/slides/HoTTEST-2022.pdf
- Midlands Graduate School talk: research/slides/MGS-2023.pdf
- CMU HoTT Seminar Talk: research/slides/CMU-2023.pdf
- Lean formalization (in progress) will be made public soon!

[AFS18] Steve Awodey, Jonas Frey, and Sam Speight.
Impredicative encodings of (higher) inductive types.
In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 76–85, 2018.

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A note on strong dinaturality, initial algebras and uniform parameterized fixpoint operators.
In *FICS*, pages 77–82, 2010.

[Wad89] Philip Wadler.
Theorems for free!
In *Proceedings of the fourth international conference on Functional programming languages and computer architecture*, pages 347–359, 1989.

Thank you!