A Crash Course on Yoneda Reasoning

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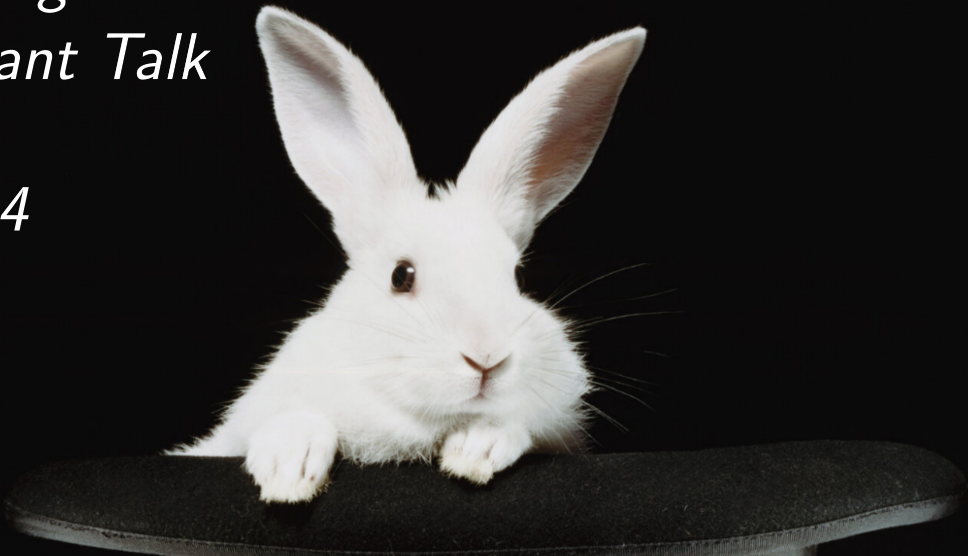

Cool Structure: Exponentials

Given two sets B, C, I can form the set C^B of all functions $B \to C$. I can define the function $\epsilon\colon C^B \times B \to C$ which sends (g, b) to $g(b)$. This satisfies the universal property of the exponential: for any $f : A \times B \to C$, there is a unique function $\overline{f} : A \to C^B$ such that

$$\begin{tikzcd}
C^B \times B \arrow[r, "\epsilon"] & C \\
A \times B \arrow[u, "\overline{f} \times \mathrm{id}_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

What about **groups** instead of sets?

Cool Structure: Exponentials

Given two groups B, C, I can form the set (group?) C^B of all homomorphisms $B \to C$. I can define the function $\epsilon \colon C^B \times B \to C$ which sends (g, b) to $g(b)$.

Does this satisfy the universal property of the exponential: for any $f : A \times B \to C$, there is a unique homomorphism $\overline{f} \colon A \to C^B$ such that

$$\begin{tikzcd}
C^B \times B \arrow[r, "\epsilon"] & C \\
A \times B \arrow[u, "\overline{f} \times \mathrm{id}_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes?

What about **groups** instead of sets?

No!

Thm. The category Grp of groups and group homomorphisms is *not* a cartesian closed category.

Lemma In a cartesian closed category $\mathbb{C}$ with an initial object $\mathbf{0}$, any morphism $\mathbb{C}(A, \mathbf{0})$ is an isomorphism.

Fact The trivial group $\mathbf{0}$ is both initial and terminal in the category of groups. So any group G has a unique morphism $\mathbb{C}(G, \mathbf{0})$.

Conclusion: Grp doesn't have exponentials

Co sn’t

Idea: Make a better version of Grp that does have these things

The category of presheaves

For any category $\mathbb{C}$, define the category **Psh**$(\mathbb{C})$ of **presheaves on** $\mathbb{C}$ to be the category whose

- Objects are functors $\mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$
- Morphisms are natural transformations.

There is a functor $\mathbf{y}\colon \mathbb{C} \to \mathbf{Psh}(\mathbb{C})$ taking each object A of $\mathbb{C}$ to the **representable presheaf** $\mathbf{y}A$.

Thm (Yoneda) For any objects A, B of $\mathbb{C}$, the morphism part of the $\mathbf{y}$ functor gives an isomorphism

$$\mathbb{C}(A, B) \cong (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, \mathbf{y}B).$$

The Yoneda Lemma: the fundamental lemma of category theory

Lemma (Yoneda) For any presheaf $F \colon \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$, there is an isomorphism

$$F\ A \cong (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, F)$$

natural in A.

Yoneda Reasoning: To define a presheaf F having a nice universal property in $\mathbf{Psh}(\mathbb{C})$,

1. Assume you already have F
2. Apply the Yoneda Lemma
3. Rewrite using the desired universal property
4. Obtain what the definition *must be*

Example 1: Products in $\mathbf{Psh}(\mathbb{C})$

Claim $\mathbf{Psh}(\mathbb{C})$ has products: for any presheaves F, G, there is a presheaf $F \times G$ such that

$$(\mathbf{Psh}(\mathbb{C}))(H, F \times G) \cong (\mathbf{Psh}(\mathbb{C}))(H, F) \times (\mathbf{Psh}(\mathbb{C}))(H, G) \qquad (*)$$

naturally in H.

By Yoneda Reasoning:

$$\begin{aligned}(F \times G)(A) &\cong (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, F \times G) && \text{(YL)}\\ &\cong (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, F) \times (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, G) && (*)\\ &\cong (F\ A) \times (G\ A) && \text{(YL)}\end{aligned}$$

Now take $(F \times G)(A) := F(A) \times G(A)$, prove this has property $(*)$.

Example 2: Exponentials in $\mathbf{Psh}(\mathbb{C})$

Claim $\mathbf{Psh}(\mathbb{C})$ has exponentials: for any presheaves F, G, there is a presheaf G^F such that

$$(\mathbf{Psh}(\mathbb{C}))(H, G^F) \cong (\mathbf{Psh}(\mathbb{C}))(H \times F, G) \qquad (**)$$

naturally in H.

By Yoneda Reasoning:

$$\begin{aligned}(G^F)(A) &\cong (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, G^F) && \text{(YL)}\\ &\cong (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A \times F, G) && (**)\end{aligned}$$

Now take $(G^F)(A) := (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A \times F, G)$, prove this has property $(**)$.

Question: Is there Yoneda machinery for verifying the definition correct?

Yes!

Already have it for representables

Want:

$$(\mathbf{Psh}(\mathbb{C}))(H, G^F) \cong (\mathbf{Psh}(\mathbb{C}))(H \times F, G)$$

Have it when $H = \mathbf{y}A$:

$$(\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, G^F) \cong G^F(A) := (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A \times F, G)(***)$$

Want: If it holds for all representables, it holds for all presheaves

The Co-Yoneda Lemma

Lemma Every presheaf H is the colimit of representable presheaves:

$$H \cong \operatorname*{colim}_{(A,a):\ \int H} \mathbf{y}A$$

$$
\begin{aligned}
(\mathbf{Psh}(\mathbb{C}))(H, G^F) &\cong (\mathbf{Psh}(\mathbb{C}))\left(\operatorname*{colim}_{(A,a) :\, \int H} \mathbf{y}A, G^F\right) && \text{(cYL)} \\
&\cong \lim_{(A,a) :\, \int H} (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A, G^F) \\
&\cong \lim_{(A,a) :\, \int H} (\mathbf{Psh}(\mathbb{C}))(\mathbf{y}A \times F, G) && (***) \\
&\cong (\mathbf{Psh}(\mathbb{C}))\left(\operatorname*{colim}_{(A,a) :\, \int H} \mathbf{y}A \times F, G\right) \\
&\cong (\mathbf{Psh}(\mathbb{C}))\left(\left(\operatorname*{colim}_{(A,a) :\, \int H} \mathbf{y}A\right) \times F, G\right) \\
&\cong (\mathbf{Psh}(\mathbb{C}))(H \times F, G) && \text{(cYL)}
\end{aligned}
$$

Summary

- Presheaf categories rich, other categories poor
- Yoneda tells you what your definitions should be
- CoYoneda helps vouch for the answer Yoneda gives

Thank you!