Towards Directed Higher Observational Type Theory

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Outline

0. Polarized and Directed CWFs
   - Turn the groupoid model into the *category model*
   - Basic components of a directed CWF
   - What synthetic (1-)category theory can look like in this setting
1. A Dream of Directed HOTT
   - Review Higher Observational Type Theory
   - Speculate recklessly

0 Polarized and Directed CWFs

The groupoidcategory model

Hofmann and Streicher defined the **groupoid model** as a *category with families* (cwf). We modify this definition to get the **category model**.

- Con = GrpdCat
- Ty(Γ) is the set of functors $\Gamma \to$ GrpdCat
- A term $M : \mathrm{Tm}(\Gamma, A)$ consists of

$$M : (\gamma : |\Gamma|) \to |A\ \gamma|$$
$$M : (\gamma_2 : \mathrm{Hom}_\Gamma(\gamma_0, \gamma_1)) \to \mathrm{Hom}_{A\ \gamma_1}(A\ \gamma_2\ (M\ \gamma_0), M\ \gamma_1)$$

- Given Γ : Con and A : Ty Γ, define the groupoidcategory $\Gamma \triangleright A$ by

$$|\Gamma \triangleright A| = \sum_{\gamma : |\Gamma|} |A\ \gamma| \qquad \mathrm{Hom}_{\Gamma \triangleright A}((\gamma_0, a_0), (\gamma_1, a_1)) = \sum_{\gamma_2 : \mathrm{Hom}_\Gamma(\gamma_0, \gamma_1)} \mathrm{Hom}_{A\ \gamma_1}(A\ \gamma_2\ a_0, a_1)$$

Idea: Internalize category-theoretic operations in the type theory

Negative Contexts and Types

The operation of *taking the opposite category* constitutes a (*covariant*) functor

$$\Gamma \xrightarrow{A} \mathsf{Cat} \xrightarrow{(-)^{\mathsf{op}}} \mathsf{Cat}$$

We can internalize this into the syntax in several ways

$$\frac{\Gamma : \mathsf{Con}}{\Gamma^- : \mathsf{Con}} \qquad \frac{A : \mathsf{Ty}\,\Gamma}{A^- : \mathsf{Ty}\,\Gamma}$$

Identity Types become Hom types

$$\frac{A : \mathrm{Ty}\,\Gamma \quad M : \mathrm{Tm}(\Gamma, A^{-}) \quad N : \mathrm{Tm}(\Gamma, A)}{M =_N M \Rightarrow N : \mathrm{Ty}\,\Gamma}$$

Problem: We can't actually do category theory with this

Want:

$$\frac{M : \mathrm{Tm}(\Gamma, A)}{\mathrm{refl}_M : \mathrm{Tm}(\Gamma, M \Rightarrow M)} \qquad \frac{F : \mathrm{Tm}(\Gamma, L \Rightarrow M) \quad G : \mathrm{Tm}(\Gamma, M \Rightarrow N)}{G \circ F : \mathrm{Tm}(\Gamma, L \Rightarrow N)}$$

But M can't be a term of both A and A^-!

or can it. . . ?

We'll now internalize the **core construction** into this type theory

$$\mathsf{Grpd} \underset{\mathsf{core}}{\overset{\perp}{\rightleftarrows}} \mathsf{Cat}$$

$$\frac{\Gamma : \mathsf{Con}}{\Gamma^0 : \mathsf{Con}} \qquad\qquad \frac{A : \mathsf{Ty}\,\Gamma}{A^0 : \mathsf{Ty}\,\Gamma}$$

$$\frac{M : \mathsf{Tm}(\Gamma, A^0)}{\langle - \rangle M : \mathsf{Tm}(\Gamma, A^-) \qquad \langle + \rangle M : \mathsf{Tm}(\Gamma, A)}$$

$$\frac{M \colon \mathsf{Tm}(\Gamma, A^0)}{\mathsf{refl}_M \colon \mathsf{Tm}(\Gamma, \langle - \rangle M \Rightarrow \langle + \rangle M)}$$

$$\frac{L : \mathsf{Tm}(\Gamma, A^-) \quad M : \mathsf{Tm}(\Gamma, A^0) \quad N : \mathsf{Tm}(\Gamma, A) \qquad F \colon \mathsf{Tm}(\Gamma, L \Rightarrow \langle + \rangle M) \quad G \colon \mathsf{Tm}(\Gamma, \langle - \rangle M \Rightarrow N)}{G \circ F \colon \mathsf{Tm}(\Gamma, L \Rightarrow N)}$$

Consequences of modelling using 1-categories

- $(P \Rightarrow Q) \quad = \quad (P \Rightarrow Q)^0 \quad = \quad (P \Rightarrow Q)^-$
- $$\frac{F_0, F_1 \colon \mathsf{Tm}(\Gamma, P \Rightarrow Q) \quad \varphi \colon \mathsf{Tm}(\Gamma, F_0 \Rightarrow= F_1)}{\varphi^{-1} \colon \mathsf{Tm}(\Gamma, F_1 \Rightarrow= F_0)}$$
- $$\frac{F_0, F_1 \colon \mathsf{Tm}(\Gamma, P \Rightarrow Q) \quad \varphi, \varphi' \colon \mathsf{Tm}(\Gamma, F_0 = F_1)}{\varphi = \varphi'}$$

Key Observation: *This* is a good setting for synthetic 1-category theory

- ✓ Objects (terms of core types) and morphisms (terms of hom-types)
- ✓ Identities and composition
- ✓ Equality between morphisms, satisfying UIP

A term $T : \mathrm{Tm}(\Gamma, A^0)$ is **terminal** if it comes equipped with

$$\begin{aligned} ! &: (N : \mathrm{Tm}(\Gamma, A^-)) \to \mathrm{Tm}(\Gamma, N \Rightarrow \langle + \rangle T) \\ !\text{-uniq} &: (N : \mathrm{Tm}(\Gamma, A^-)) \to (H : \mathrm{Tm}(\Gamma, N \Rightarrow \langle + \rangle T)) \to \mathrm{Tm}(\Gamma, !N = H) \end{aligned}$$

Conjecture/WIP We can equip the category model with (co- and contravariant) dependent types, allowing us to internalize these dependent functions.

Current/Future work

- Negative context extension: for $B : \mathsf{Ty}\ \Gamma^-$, we have $(\Gamma \triangleright^- B) : \mathsf{Con}$
- Abstract definition of **polarized CWFs** and **directed CWFs**
  - Polarized: CWF with $\pm$ contexts, $\pm$ types, and $\pm$ context extension
  - Directed: Polarized CWF, with core contexts & types, polarized dependent types, and hom types (and possibly universes)
- Other examples: preorder model (with setoids as the core), poset model, univalent categories?
- Syntax model and initiality proof

1 A Dream of Directed HOTT

Key Idea of HOTT

- Define identity types observationally
  - Definitional Univalence

Consider

$$\frac{z\colon A \vdash Q(z)\colon \mathbf{U} \qquad p\colon a =_A a'}{\mathsf{ap}_{z.Q}(p)\colon Q(a) =_{\mathbf{U}} Q(a')}$$

Definitional Univalence

$$T =_{\mathbf{U}} T' \qquad \equiv \qquad \sum_{R\colon T \to T' \to \mathbf{U}} \text{is-1-to-1}(R)$$

$$\text{is-1-to-1}(R) :\equiv \left(\prod_{t:T} \mathsf{is\_contr}\left(\sum_{t':T'} R\ t\ t'\right)\right) \times \left(\prod_{t':T'} \mathsf{is\_contr}\left(\sum_{t:T} R\ t\ t'\right)\right)$$

Dependent/heterogeneous identity types: for $u\colon Q(a)$ and $v : Q(a')$,

Identity Types of Σ-types

$$p \colon a =_A a'$$
$$\mathsf{ap}_{z.Q}(p) \colon Q(a) =_{\mathbf{U}} Q(a')$$
$$\pi_1(\mathsf{ap}_{z.Q}(p)) \colon Q(a) \to Q(a') \to \mathbf{U}$$
$$u =^{p}_{z.Q} v \equiv (\pi_1(\mathsf{ap}_{z.Q}(p)))\; u\; v$$

Defn.

$$(a, u) = (a', v) \qquad \equiv \qquad \sum_{p \colon a =_A a'} u =^{p}_{z.Q} v$$

Now make it directed!

$$\frac{z\colon A^- \vdash Q(z)\colon \mathbf{U} \qquad p\colon a \Rightarrow a'}{\mathrm{am}_{-z.Q}(p)\colon Q(a') \Rightarrow_{\mathbf{U}} Q(a)}$$

Directed Definitional Univalence

$$T \Rightarrow_{\mathbf{U}} T' \qquad \equiv \qquad \sum_{R\colon T \to T' \to \mathbf{U}} \left(\prod_{t\colon T} \mathrm{is\_contr} \left(\sum_{t'\colon T'} R\ t\ t' \right) \right)$$

Dependent/heterogeneous hom types: for $u\colon Q(a)$ and $v : Q(a')$,

$$u \Rightarrow^{p}_{-z.Q} v \qquad :\equiv \qquad (\pi_1(\mathrm{am}_{-z.Q}(p)))\ v\ u$$

Identity Types of contravariant Σ-types

$$
\begin{aligned}
p \colon a &\Rightarrow a' \\
\mathrm{am}_{-z.Q}(p) \colon Q(a') &\Rightarrow_{\mathbf{U}} Q(a) \\
\pi_1(\mathrm{am}_{-z.Q}(p)) \colon Q(a') &\to Q(a) \to \mathbf{U} \\
u \Rightarrow^{p}_{-z.Q} v &\equiv (\pi_1(\mathrm{am}_{-z.Q}(p)))\ v\ u
\end{aligned}
$$

Defn.

$$(a, u) \Rightarrow (a', v) \qquad \equiv \qquad \sum_{p \colon a \Rightarrow a'} u \Rightarrow^{p}_{-z.Q} v$$

Synthetic slice categories?

Consider

$$z\colon A^0 \vdash \langle - \rangle z \Rightarrow N$$

for some fixed $N\colon A$. A natural choice would then be to put

$$\pi_1(\mathsf{am}_{-z.z\Rightarrow N}(p))\ \varphi'\ \varphi :\equiv \quad \varphi = \varphi' \circ p$$

So then if we define

$$A/N :\equiv \sum_{a\colon A^0} \langle - \rangle a \Rightarrow N$$

we get:

$$(a, \varphi) \Rightarrow (a', \varphi') \qquad \equiv \qquad \sum_{p\colon \langle - \rangle a \Rightarrow \langle + \rangle a'} \varphi = \varphi' \circ p$$

Things that need to be worked out (not exhaustive...)

- Directed presheaf model
  - Polarized CWF
  - Core types
  - Co- and contra-variant dependent types
  - Directed analogue of *Lifting Grothendieck Universes*
- Polarized telescopic HOAS
- Hom types in directed presheaf model, properly define am with respect to arbitrary telescopes
- Identity types between homs
- Give proper statement of directed definitional univalence
- Clarify what it means to use core types in a negative telescope
- Various other semantic concerns (like the ones that arise in undirected HOTT?)

Thank you!