Presheaf Models of Polarized Higher-Order Abstract Syntax

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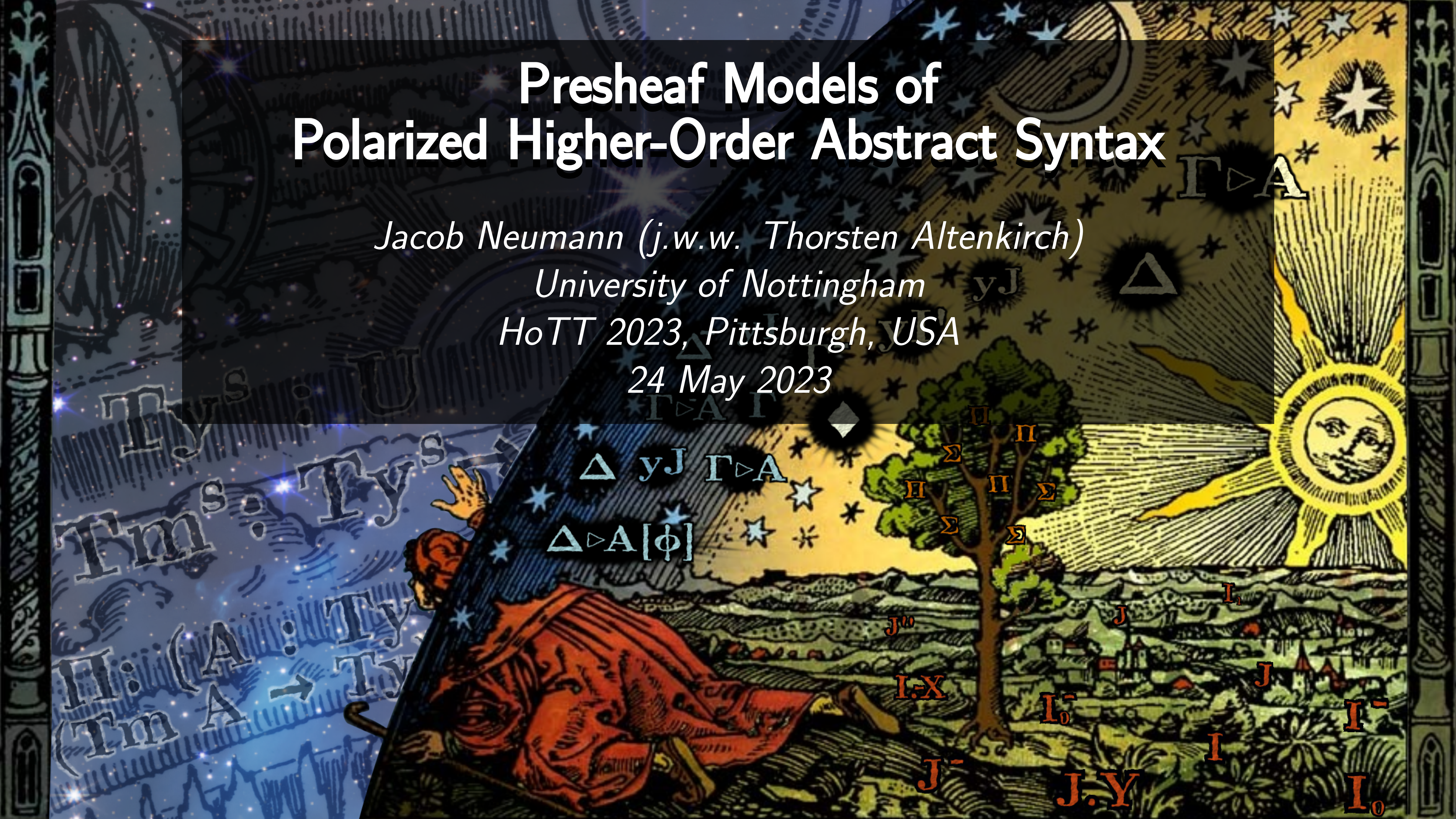


Where to find more detail

jacobneu.github.io/research/directedTT/landing.html

What I'm interested in:

Directed TT

$\times$

Higher Observational TT?

Key Component :
HOAS with polarities

0 Polarized Type Theory

Our approach to type theory: Semantics first!

Defn. A **category with families (CwF)** is a (generalized) algebraic structure, consisting of:

- A category Con of *contexts* and *substitutions*, with a terminal object $\bullet$, the *empty context*
- A presheaf $\mathsf{Ty} \colon \mathsf{Con}^{\mathsf{op}} \to \mathsf{Set}$ of *types*
- A presheaf $\mathsf{Tm} \colon (\int \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$ of *terms*
- An operation of *context extension*:

$$\frac{\Gamma \colon \mathsf{Con} \quad A : \mathsf{Ty}\ \Gamma}{\Gamma \triangleright A : \mathsf{Con}}$$

so that $\Gamma \triangleright A$ is a 'locally representing object' (in the sense spelled out on the next slide)

The Local Representability Condition

For any I, J : Con and any J : Ty Γ,

$$\mathrm{Con}(I, J \triangleright Y) \cong \sum_{j\,:\,\mathrm{Con}(I,J)} \mathrm{Tm}(I, Y[j])$$

natural in I.

Three Important Models of Type Theory

Set

The Set Model

[Dyb95, Hof97]

- Contexts are **sets**
- Types in context Γ are families of **sets** over Γ

Setoid

The Setoid Model

[Hof94, Alt99]

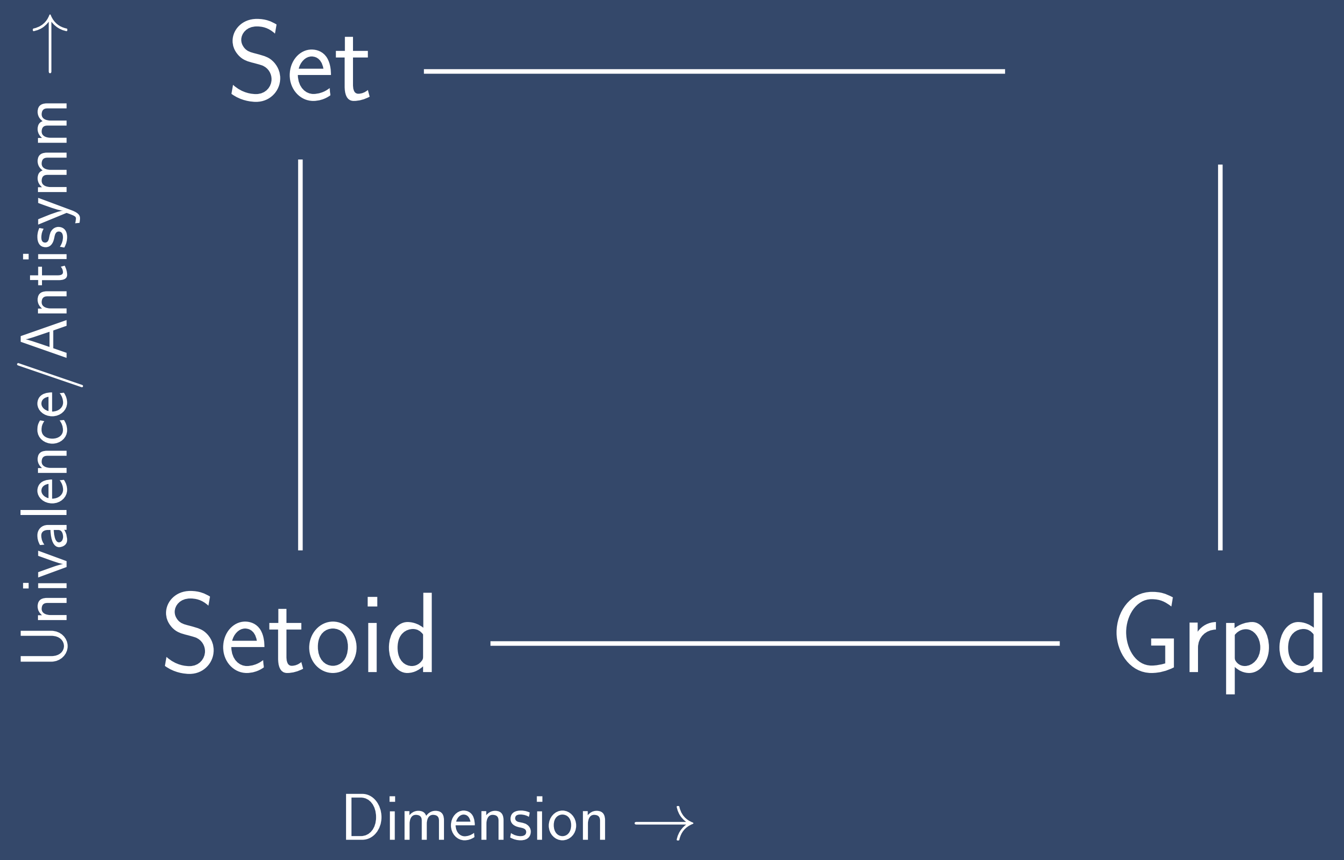
- Contexts are **setoids**
- Types in context Γ are families of **setoids** over Γ

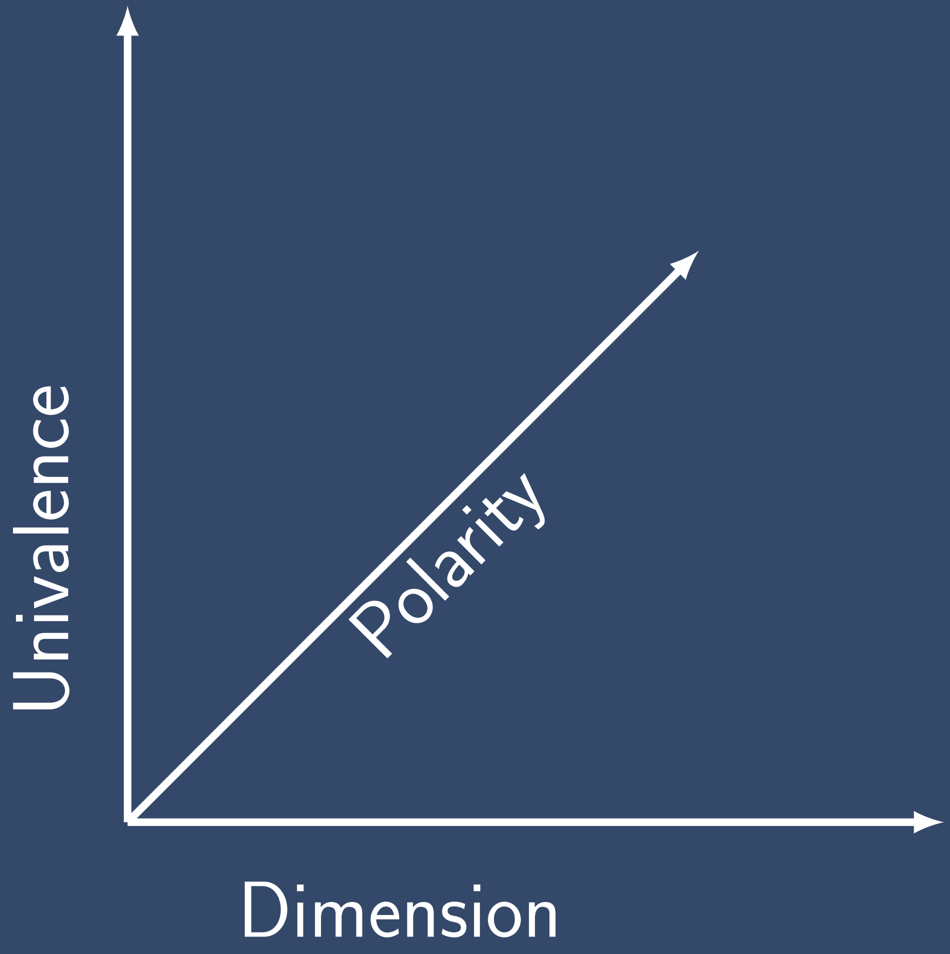
Grpd

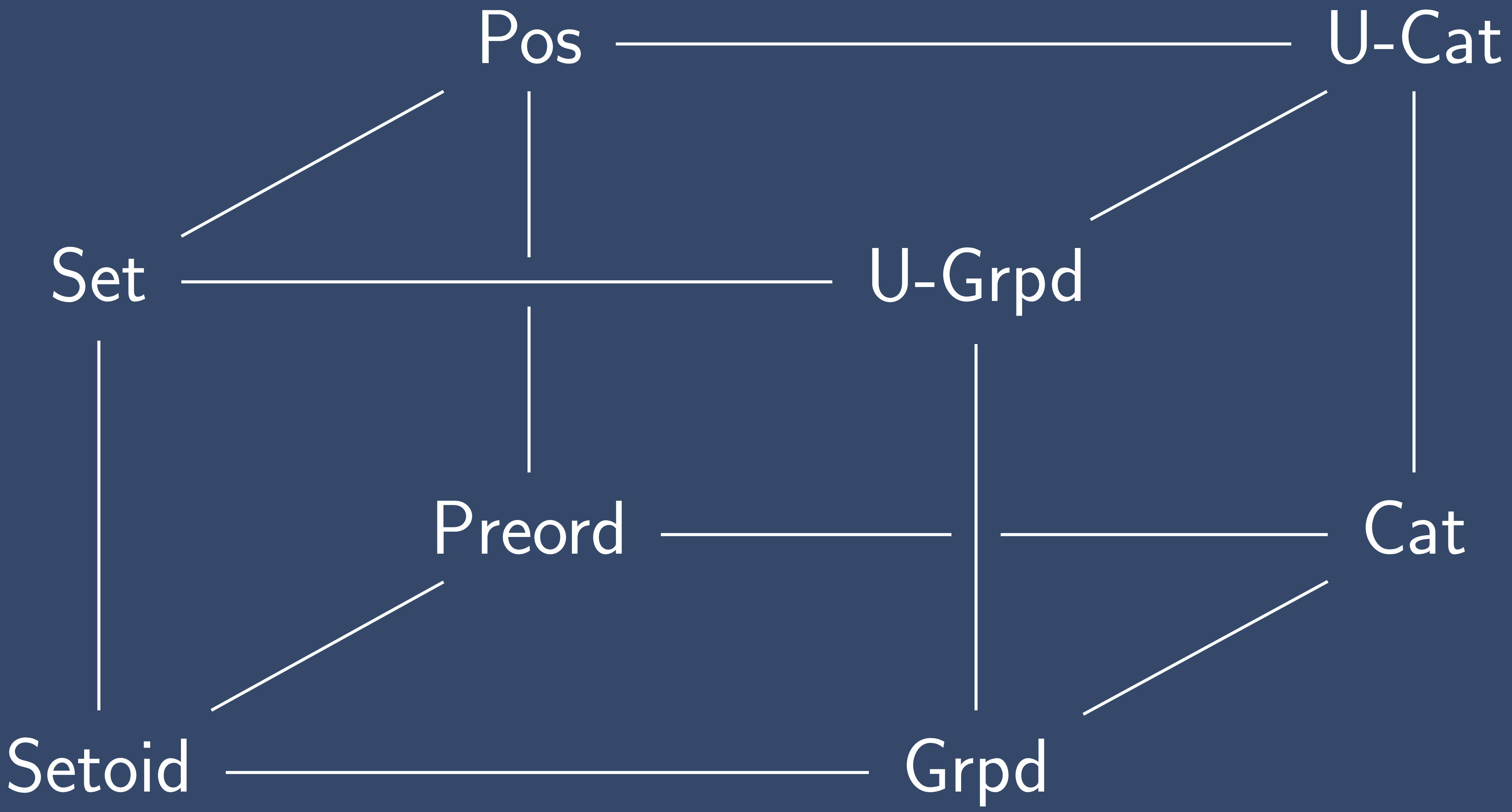
The Groupoid Model

[HS95]

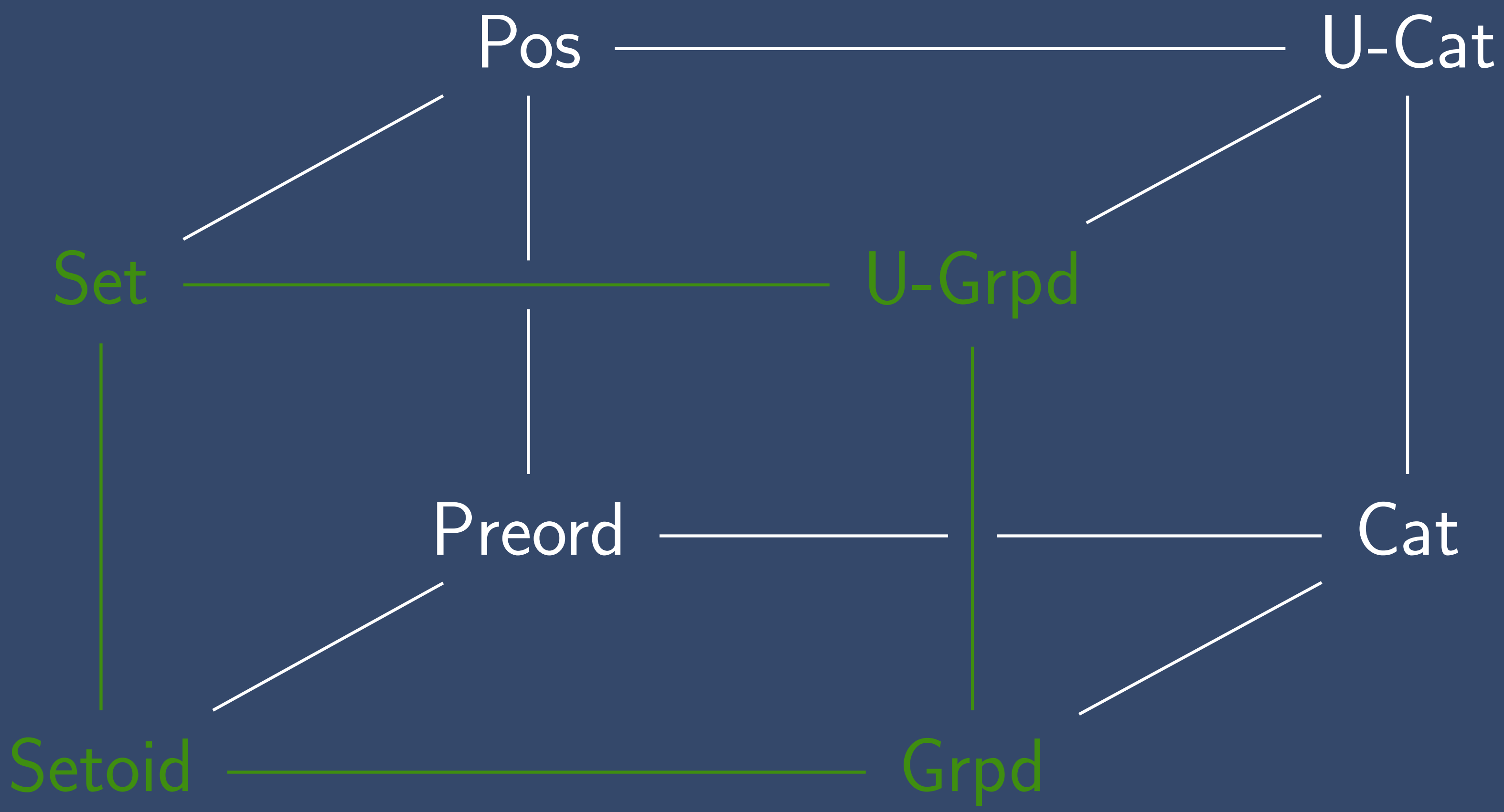
- Contexts are **groupoids**
- Types in context Γ are families of **groupoids** over Γ


Set
Setoid
Grpd
Univalence/Antisymm →
Dimension →


Univalence
Polarity
Dimension

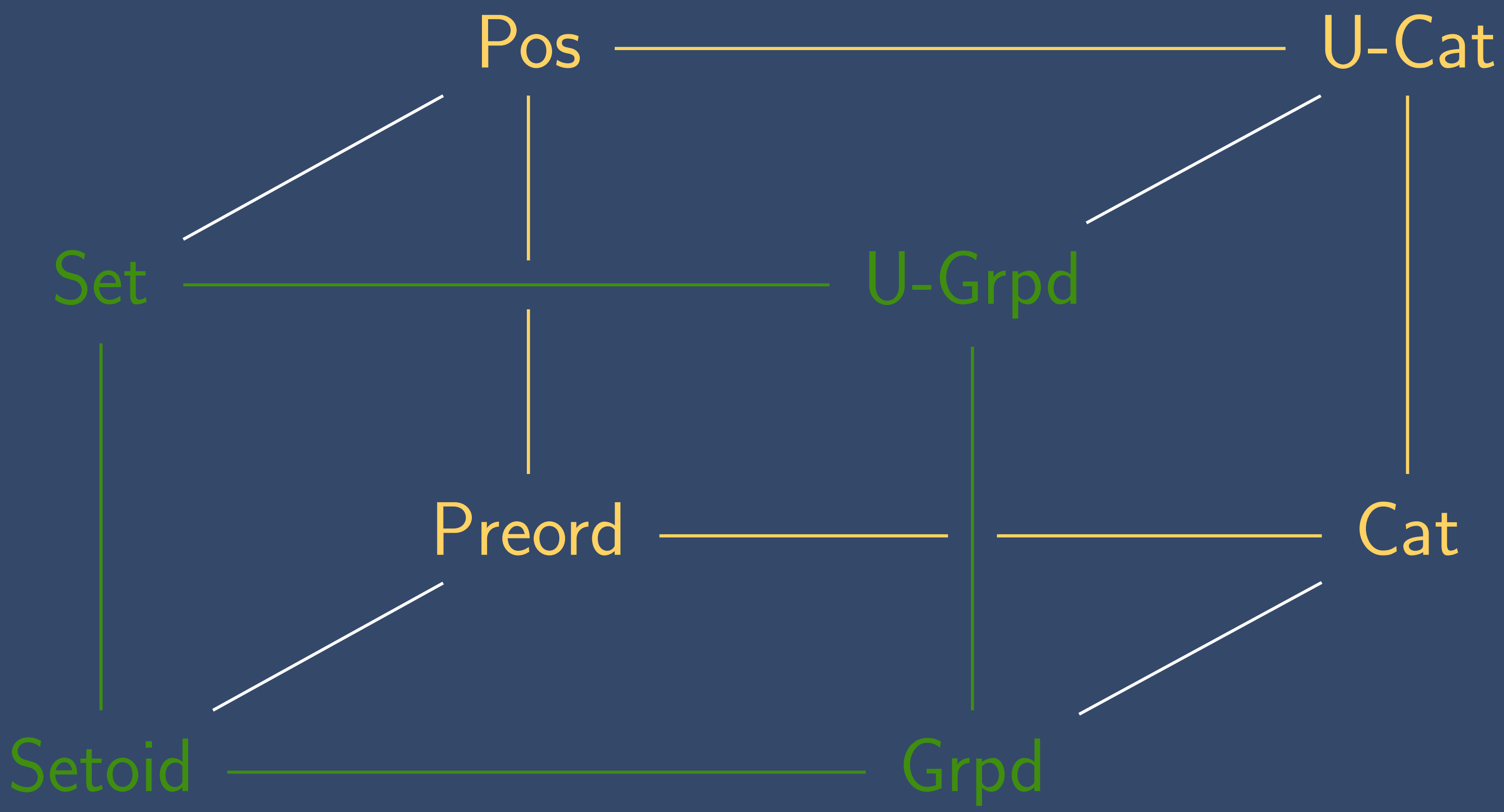


Pos
U-Cat
Set
U-Grpd
Preord
Cat
Setoid
Grpd



CwFs

What kinds of models have the back-face structures as contexts?



CwFs PCwFs

What is a polarized CwF?

A **(concrete) polarized category with families** is a (generalized) algebraic structure, consisting of:

- Con, $\bullet$, Ty, Tm as in the definition of CwF
- A functor $(\_)^{-}$: Con $\to$ Con such that $(J^{-})^{-} = J$ and $\bullet^{-} = \bullet$
- For each J: Con, a function $(\_)^{-}$: Ty $J \to$ Ty J such that $(Y^{-})^{-} = Y$
- Two operations of *context extension*: for s either $+$ or $-$,

$$\frac{J : \text{Con} \qquad Y : \text{Ty}(J^{s})}{J \triangleright^{s} Y : \text{Con}}$$

The Local Representability Condition

For any I, J : Con and any J : Ty Γ^s,

$$\mathsf{Con}(I, J \triangleright^s Y) \quad \cong \quad \sum_{j\,:\,\mathsf{Con}(I,J)} \mathsf{Tm}(I^s, Y[j^s]^s)$$

natural in I.

The Category Interpretation of Type Theory

The category model of type theory is a PCwF where

- Con is the category of categories and functors
- Ty J is the set of J-indexed families of categories (i.e. pseudofunctors $J \to \mathsf{Cat}$)
- $\ldots$
- The context negation functor is the operation of taking **opposite categories**, which extends to a functor $\mathsf{Cat} \to \mathsf{Cat}$
- Type negation is given by post-composition with the opposite category functor

Context Extension in the Category Model

$$\frac{J : \mathrm{Con} \qquad Y : \mathrm{Ty}(J^s)}{J \triangleright^s Y : \mathrm{Con}} \qquad (s = +, -)$$

$$|J \triangleright^s Y| = \sum_{j : |J|} |Y\ j|$$

$$\mathrm{Hom}_{J \triangleright^+ Y}((j_0, y_0), (j_1, y_1)) = \sum_{j_2 : \mathrm{Hom}(j_0, j_1)} \mathrm{Hom}_{Y(j_1)}(Y\ j_2\ y_0, y_1)$$

$$\mathrm{Hom}_{J \triangleright^- Y}((j_0, y_0), (j_1, y_1)) = \sum_{j_2 : \mathrm{Hom}(j_0, j_1)} \mathrm{Hom}_{Y(j_0)}(y_0, Y\ j_2\ y_1)$$

Polarized Pi Types

The category model, the preorder model, etc. admit the polarized Π-types of [LH11]:

$$\frac{Y : \mathrm{Ty}\ J^{-} \qquad Z : \mathrm{Ty}(J \triangleright^{-} Y)}{\Pi\ Y\ Z : \mathrm{Ty}\ J}$$

$$\frac{M : \mathrm{Tm}(J \triangleright^{-} Y, Z)}{(\lambda\ M) : \mathrm{Tm}(J, \Pi\ Y\ Z)}$$

$$\frac{M : \mathrm{Tm}(J, \Pi\ Y\ Z) \qquad N : \mathrm{Tm}(J^{-}, Y^{-})}{(M\ N) : \mathrm{Tm}(J, Z[\overline{N}])}$$

1 Presheaf Semantics of HOAS

Need to explicitly require stability under substitution

Definition 3.15 A CwF supports Π-types if for any two types $\sigma \in Ty(\Gamma)$ and $\tau \in Ty(\Gamma.\sigma)$ there is a type $\Pi(\sigma,\tau) \in Ty(\Gamma)$ and for each $M \in Tm(\Gamma.\sigma,\tau)$ there is a term $\lambda_{\sigma,\tau}(M) \in Tm(\Gamma,\Pi(\sigma,\tau))$ and for each $M \in Tm(\Gamma,\Pi(\sigma,\tau))$ and $N \in Tm(\Gamma,\sigma)$ there is a term $App_{\sigma,\tau}(M,N) \in Tm(\Gamma,\tau\{\overline{M}\})$ such that (the appropriately typed universal closures of) the following equations hold:

$$\begin{array}{lcll}
App_{\sigma,\tau}(\lambda_{\sigma,\tau}(M),N) & = & M\{\overline{N}\} & \Pi\text{-C} \\
\Pi(\sigma,\tau)\{f\} & = & \Pi(\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}) \in Ty(\mathrm{B}) & \Pi\text{-S} \\
\lambda_{\sigma,\tau}(M)\{f\} & = & \lambda_{\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}}(M\{\mathsf{q}(f,\sigma)\}) & \lambda\text{-S} \\
App_{\sigma,\tau}(M,N)\{f\} & = & App_{\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}}(M\{f\},N\{f\}) & App\text{-S}
\end{array}$$

} annoying! (for Π-S, λ-S, App-S)

From [Hof97, 3.3]

Solution: Use higher-order abstract syntax!

(and interpret it in a presheaf category!)

Steps

1. Presheaf Model
2. Lift Grothendieck Universe(s) [HS99]
3. Higher-Order Abstract Syntax [Hof99]

For a fixed (small) category $\mathbb{C}$, we can define the **presheaf model** (over $\mathbb{C}$) to be a CwF $(\widehat{\mathsf{Con}}, \widehat{\mathsf{Ty}}, \widehat{\mathsf{Tm}}, \ldots)$, where

- Contexts are presheaves $\mathbb{C}^{\mathsf{op}} \to \mathsf{Set}$
- Substitutions are natural transformations
- Types in context Γ are presheaves on $\int \Gamma$
- The empty context $\blacklozenge$ is the constant-$\mathbb{1}$ presheaf
- $\ldots$

Claim This model of type theory supports Π-types

We want a universe, i.e. a closed type $\mathbf{U}$ such that

$$\widehat{\mathsf{Tm}}(\Gamma, \mathbf{U}) \cong \widehat{\mathsf{Ty}}\,\Gamma$$

Thankfully, we're in a presheaf category and can do Yoneda calculations:

$$\begin{aligned} \mathbf{U}\ I &\cong \widehat{\mathsf{Con}}(\mathbf{y}\ I, \mathbf{U}) \\ &\cong \widehat{\mathsf{Tm}}(\mathbf{y}\ I, \mathbf{U}) \\ &\cong \widehat{\mathsf{Ty}}(\mathbf{y}\ I) \\ &= \mathsf{Set}^{(\int \mathbf{y}\ I)^{\mathsf{op}}} \\ &= \mathsf{Set}^{(\mathbb{C}/I)^{\mathsf{op}}} \end{aligned}$$

So just define $\mathbf{U}\ I$ to be the set of presheaves on $\mathbb{C}/I$.

What if $\mathbb{C}$ is *itself* a CwF?

Key Idea: Talk about the "ground" CwF structure using the presheaf CwF structure

Higher-Order Abstract Syntax

Semantics	**HOAS**
$\mathsf{Ty} \colon \mathbb{C}^{\mathsf{op}} \to \mathsf{Set}$	$\mathsf{Ty} \colon \mathbf{U}$
$\mathsf{Tm} \colon (\int \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$	$\mathsf{Tm} \colon \mathsf{Ty} \to \mathbf{U}$
$\ldots$	$\Pi \colon (A \colon \mathsf{Ty}) \to (\mathsf{Tm}\ A \to \mathsf{Ty}) \to \mathsf{Ty}$

2 Polarized HOAS

Problem: How do we talk about operations on contexts, after we've abstracted them away?

Hint: we don't need context- and type-negation to be independent

$$\mathsf{Con}(I, J \triangleright^{s} Y) \quad \cong \quad \sum_{j\,:\,\mathsf{Con}(I,J)} \mathsf{Tm}(I^{s}, Y[j^{s}]^{s})$$

$$\frac{M : \mathsf{Tm}(J, \Pi\ Y\ Z) \quad N : \mathsf{Tm}(J^{-}, Y^{-})}{(M\ N) : \mathsf{Tm}(J, Z[\overline{N}])}$$

Defining Ty^-

$$\begin{aligned}
\mathsf{Ty}^- &: \mathsf{Con}^{\mathsf{op}} \to \mathsf{Set} \\
\mathsf{Ty}^- J &:= \mathsf{Ty}(J^-) \\
Y[j] &:= Y[j^-] && (j : \mathsf{Con}(I, J),\ Y : \mathsf{Ty}^-\ J)
\end{aligned}$$

$$\begin{aligned}
\mathsf{Tm}^- &: \textstyle\int \mathsf{Ty}^- \to \mathsf{Set} \\
\mathsf{Tm}^-(J, Y) &:= \mathsf{Tm}(J^-, Y^-) \\
M[j] &:= M[j^-] && (j : \mathsf{Con}(I, J),\ M : \mathsf{Tm}^-(J, Y))
\end{aligned}$$

Revisited: we don't need context- and type-negation to be independent

$$\mathsf{Con}(I, J \triangleright^s Y) \quad \cong \quad \sum_{j\,:\,\mathsf{Con}(I,J)} \mathsf{Tm}^s(I, Y[j])$$

$$\frac{M : \mathsf{Tm}(J, \Pi\ Y\ Z) \qquad N : \mathsf{Tm}^-(J, Y)}{(M\ N) : \mathsf{Tm}(J, Z[\overline{N}])}$$

Defn. An **abstractly polarized CwF** is a category Con with a terminal object $\bullet$ and *two* CwF structures

$$\mathrm{Ty}, \mathrm{Tm}, \triangleright \qquad \text{and} \qquad \mathrm{Ty}^{-}, \mathrm{Tm}^{-}, \triangleright^{-}$$

Question What more should be added to this definition?

- $\mathrm{Ty}\ \bullet = \mathrm{Ty}^{-}\ \bullet$
- ??

This seems to be the right approach

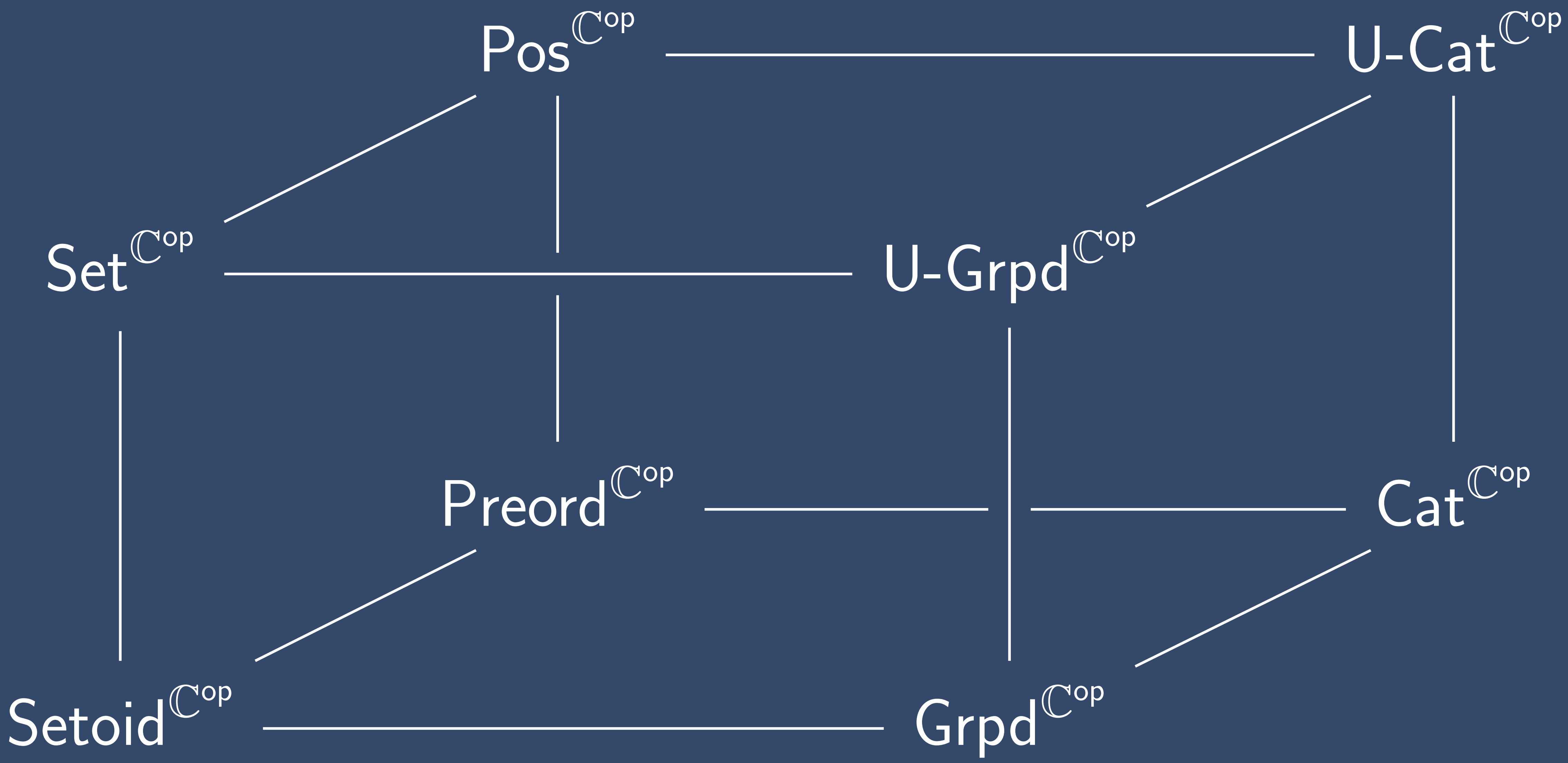
- Better fits the formulation of CwFs as natural models [Awo18]
- When adapting [ABK+21]'s Agda formalization of the setoid model, it is very straightforward to define it as an abstract PCwF but proving much more difficult to do as a concrete PCwF

Idea The presheaf model over an abstract PCwF gives us a 2-level type theory: the inner layer polarized, the outer unpolarized

Semantics	**HOAS**
$\mathrm{Ty}^s : \widehat{\mathrm{Tm}}(\blacklozenge, \mathbf{U})$	$\mathrm{Ty}^s : \mathbf{U}$
$\mathrm{Tm}^s : \widehat{\mathrm{Tm}}(\blacklozenge, \mathrm{Ty}^s \Rightarrow \mathbf{U})$	$\mathrm{Tm}^s : \mathrm{Ty}^s \to \mathbf{U}$
$\ldots$	$\Pi : (A : \mathrm{Ty}^-) \to (\mathrm{Tm}^- \, A \to \mathrm{Ty}) \to \mathrm{Ty}$

Further Topics of Study

- Core types, neutral-zoned contexts
- Hom types
- Polarized telescopes
- Directed Observational TT
- Formalization
- Connections to other varieties of polarized/directed TT
- Polarizing both layers


Pos^ℂop
U-Cat^ℂop
Set^ℂop
U-Grpd^ℂop
Preord^ℂop
Cat^ℂop
Setoid^ℂop
Grpd^ℂop

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Thank you!!