Paranatural Category Theory

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Question of the day: Where have I seen this before?

Table of Contents

Sources

- Collection of links: jacobneu.github.io/research/paranat
- arXiv preprint: arxiv.org/abs/2307.09289
- HoTTEST talk:
  - Video: youtube.com/watch?v=X4v5HnnF2-o
  - Slides: research/slides/HoTTEST-2022.pdf
- Midlands Graduate School talk: research/slides/MGS-2023.pdf
- Lean formalization (in progress) will be made public soon!

0 Category-Theoretic Church Numerals

Encoding $\mathbb{N}$

$$\overline{n} \quad :\equiv \quad \lambda f.f^n$$

What kind of thing is $\overline{n}$?

$$\overline{n} \quad : \quad \mathrm{Hom}(I, I) \to \mathrm{Hom}(I, I)(\overline{n})_I \quad : \quad \mathrm{Hom}(I, I)$$

Goal Articulate a condition on this data, such that

- "Soundness": every $\overline{n}$ satisfies it
- "Completeness": you can prove the η law for $\mathbb{N}$ from it

Is $\overline{n}$ a endo-natural transform of Hom?

$\mathrm{Hom} : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$, "Hom is a difunctor on $\mathbb{C}$"

So a natural transformation $\mathrm{Hom} \to \mathrm{Hom}$ would have components indexed by $\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$.

Requires Too Much

Requires Too Little

Same problem with extranaturality

Let $F\colon A \times B \times B^{\mathrm{op}} \to D$ and $G\colon A \times C \times C^{\mathrm{op}} \to D$ be functors. A family of morphisms

$$\alpha_{a,b,c}\colon F(a,b,b) \to G(a,c,c)$$

for $a \in A$, $b \in B$, and $c \in C$ is said to be ***natural***, or more precisely *ordinary-natural in a and extranatural in b and c*, if the following hold.

Requires Too Much

Requires Too Little

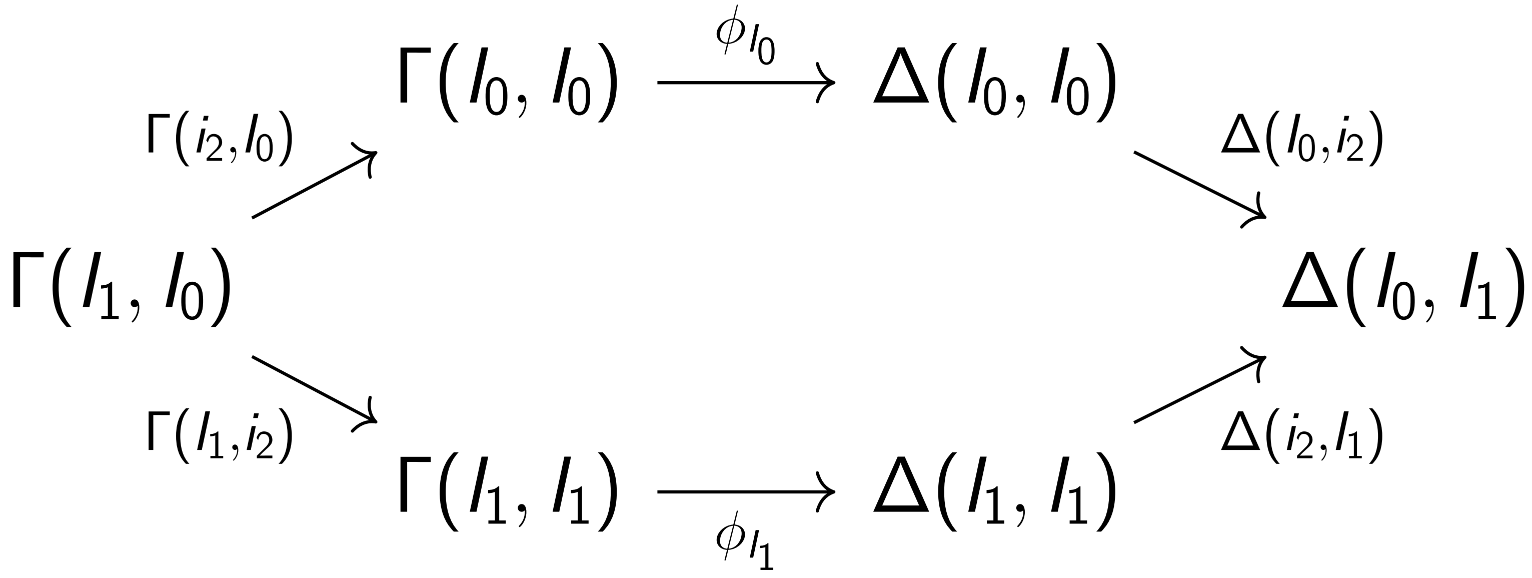
We need a *diagonal* notion of transformation

Dinatural Transformations

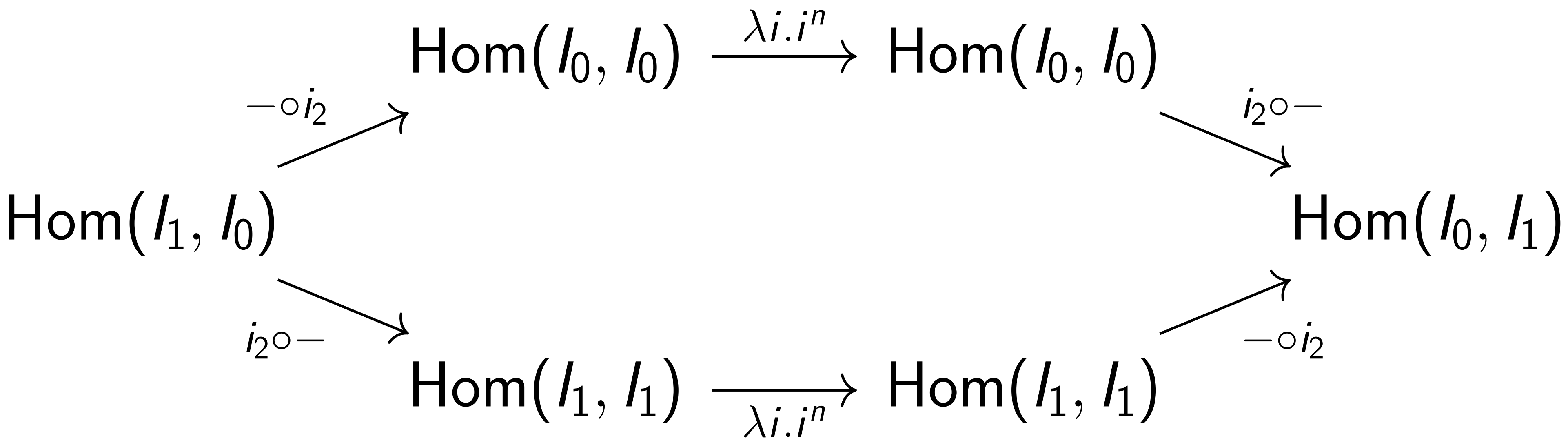
For $\Gamma, \Delta : \mathbb{C}^{op} \times \mathbb{C} \to \mathrm{Set}$, a **dinatural transformation** from Γ to Δ is a family of maps

$$\phi_I \quad : \quad \Gamma(I, I) \to \Delta(I, I)$$

indexed by objects I of $\mathbb{C}$, such that, for every $i_2 \colon \mathrm{Hom}(I_0, I_1)$, the following hexagon commutes.

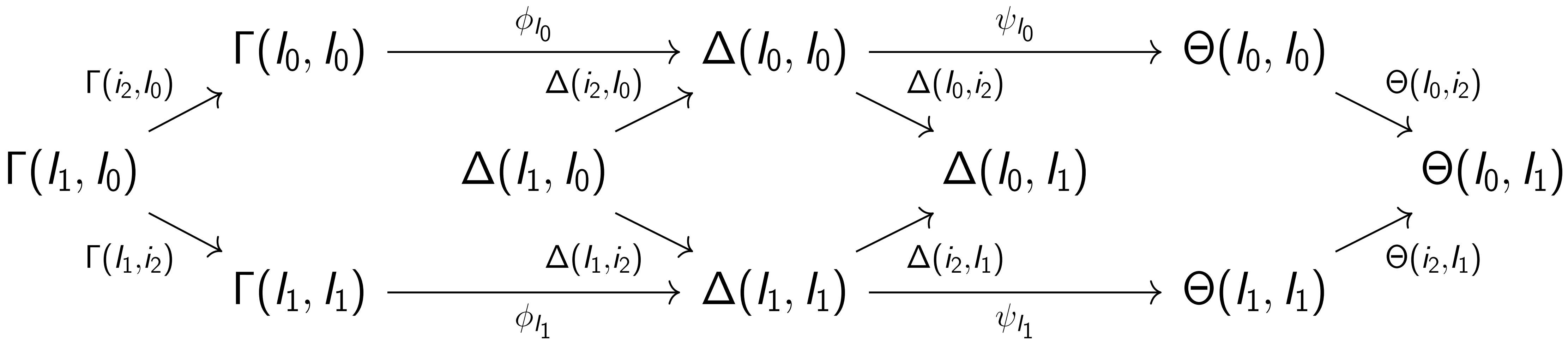


Endo-dinatural Transformations of Hom



$$\text{for all } i_2 : \mathrm{Hom}(I_0, I_1) \text{ and all } i_2' : \mathrm{Hom}(I_1, I_0),$$
$$i_2 \circ (i_2' \circ i_2)^n = (i_2 \circ i_2')^n \circ i_2$$

- Dinaturality condition doesn't seem to be saying anything worthwhile
  - No hope of proving η
- Dinaturals don't compose
  - $\overline{m \cdot n} = \overline{m} \circ \overline{n}$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Problems with dinaturality

- Dinaturality condition doesn't seem to be saying anything worthwhile
  - No hope of proving η
- Dinaturals don't compose
  - $\overline{m \cdot n} = \overline{m} \circ \overline{n}$

Requires Too Much

Requires Too Little

For $\Gamma, \Delta : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$, a **paranatural transformation** (known as a **strong dinatural transformation** in the literature) from Γ to Δ is a family of maps

$$\phi_I \quad : \quad \Gamma(I, I) \to \Delta(I, I)$$

indexed by objects I of $\mathbb{C}$, such that, for every $g_0 \colon \Gamma(I_0, I_0)$, $g_1 \colon \Gamma(I_1, I_1)$ and $i_2 : \mathrm{Hom}(I_0, I_1)$ such that

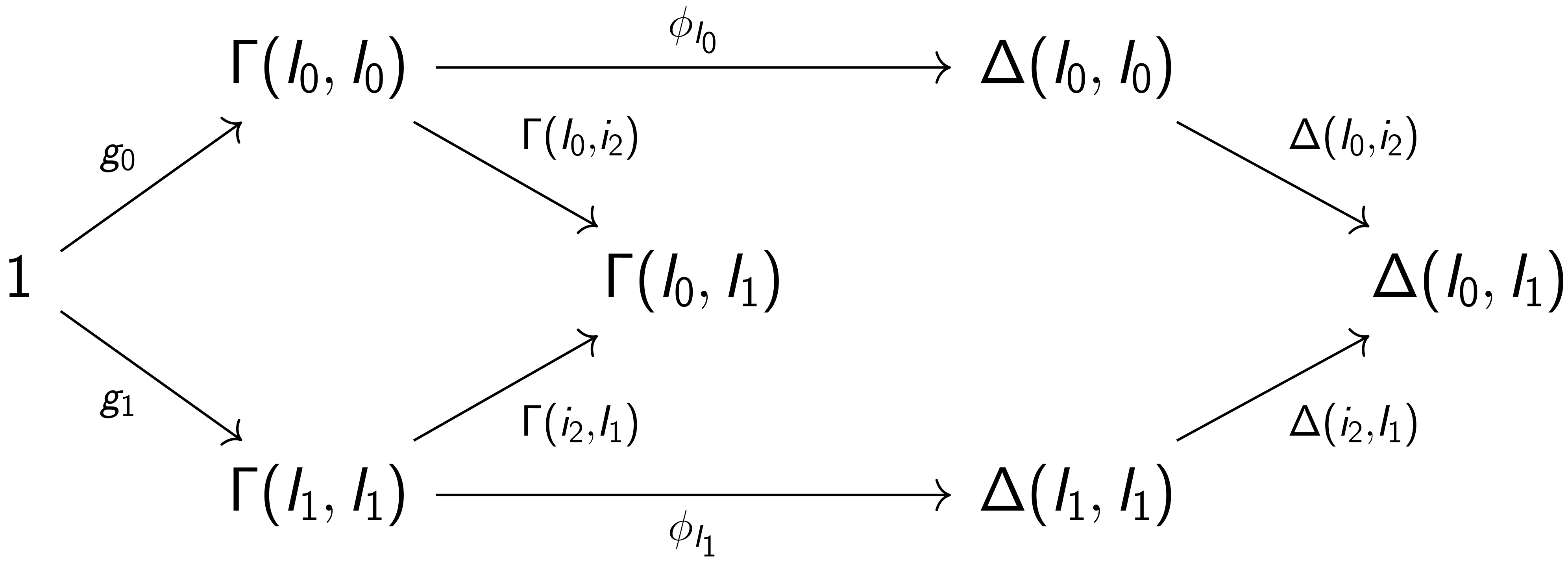
$$\Gamma(I_0, i_2)\; g_0 \quad = \quad \Gamma(i_2, I_1)\; g_1$$

it is the case that

$$\Delta(I_0, i_2)\; (\phi_{I_0}\; g_0) \quad = \quad \Delta(I_1, i_2)\; (\phi_{I_1}\; g_1).$$

Notation Write $\phi \colon \Gamma \xrightarrow{\diamond} \Delta$ to mean that ϕ is a paranatural transformation from Γ to Δ.

if the diamond commutes, so does the hexagon



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Paranaturals are nice

- ✓ Composition
- ✓ η
- ✓ When both the domain & codomain are functors (or both presheaves), paranaturality coincides with usual naturality

1 Parametric Polymorphism

$$\texttt{rev}\colon \mathsf{List}\ \mathbb{N} \to \mathsf{List}\ \mathbb{N}$$

$$\texttt{rev}\colon \mathsf{List}\ \mathbf{2} \to \mathsf{List}\ \mathbf{2}$$

$$\texttt{rev}\colon \mathsf{List}\ \texttt{string} \to \mathsf{List}\ \texttt{string}$$

$$\texttt{rev}\colon \mathsf{List}(\mathsf{List}\ \mathbb{N}) \to \mathsf{List}(\mathsf{List}\ \mathbb{N})$$

$$\vdots$$

$$\texttt{rev}\colon \forall\alpha.\mathsf{List}\ \alpha \to \mathsf{List}\ \alpha$$

Key Idea: A polymorphic function cannot examine or case on α

Parametricity

The topic of **parametricity** is the precise statement of what “cannot examine or case on α” means. This was done by Reynolds, using logical relations.

For the type $\forall\alpha.\mathsf{List}\ \alpha \to \mathsf{List}\ \alpha$, this ends up being just naturality

$$\begin{array}{ccc} \mathsf{List}\ X & \xrightarrow{\texttt{rev}} & \mathsf{List}\ X \\ {\scriptstyle \mathsf{map}\ f}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{map}\ f} \\ \mathsf{List}\ Y & \xrightarrow[\texttt{rev}]{} & \mathsf{List}\ Y \end{array}$$

Problem:

Parametricity=Naturality doesn't extend to difunctors

$$\texttt{sort}\colon \forall\alpha.(\alpha \times \alpha \to \mathbf{2}) \to \mathsf{List}\ \alpha \to \mathsf{List}\ \alpha$$

"Free theorem" for this type (Wadler):

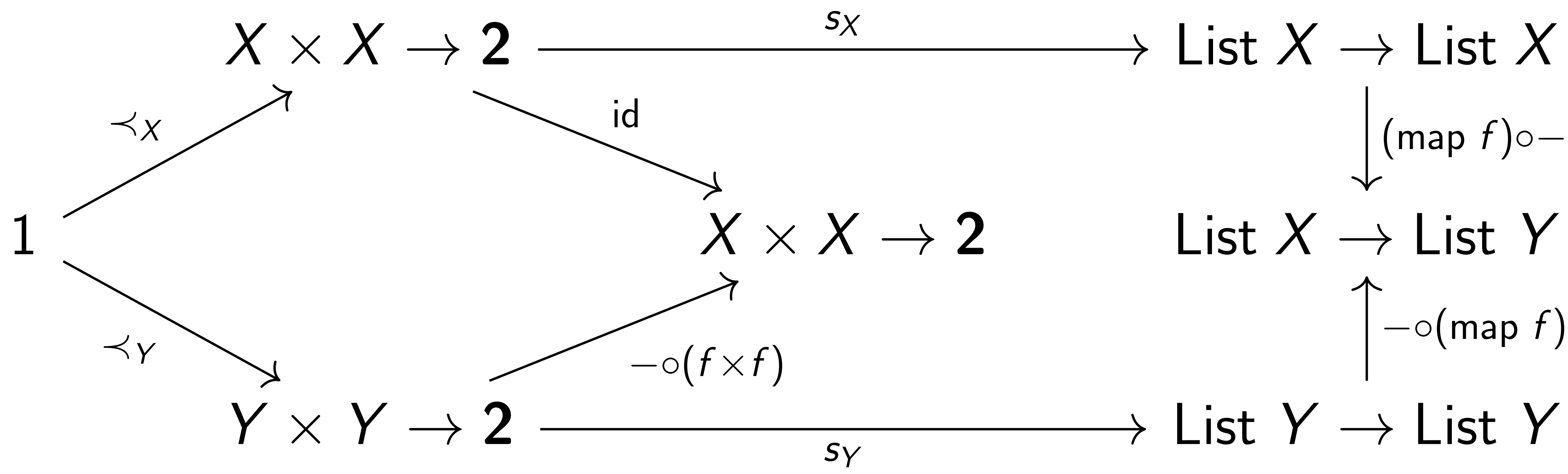
if $\prec_X\colon X \times X \to \mathbf{2}$ and $\prec_Y\colon Y \times Y \to \mathbf{2}$ and $f\colon X \to Y$ is such that

$$(x \prec_X x') \quad = \quad (f(x) \prec_Y f(x')) \qquad \text{for all } x, x' : X$$

then, for any $s\colon \forall\alpha.(\alpha \times \alpha \to \mathbf{2}) \to \mathsf{List}\ \alpha \to \mathsf{List}\ \alpha$

$$(\mathsf{map}\ f) \circ (s_X\ \ \prec_X) = (s_Y\ \ \prec_Y) \circ (\mathsf{map}\ f).$$

if the diamond commutes, so does the hexagon



General Conjecture:

Paranaturality captures the correct intuitions for parametric polymorphism

2 Basic Paranatural Category Theory

Prop. For a $|\mathbb{C}|$-indexed family of maps $\phi_I : \Gamma(I, I) \to \Delta(I, I)$, the following are equivalent:

(1) ϕ is a paranatural transformation

(2) For all $i_2 : \mathrm{Hom}(I_0, I_1)$,

$$\Delta(I_0, i_2) \circ \phi_{I_0} \circ p_0 = \Delta(i_2, I_1) \circ \phi_{I_1} \circ p_1$$

where p_0, p_1 are the projection maps of the pullback of $\Gamma(I_0, i_2)$ along $\Gamma(i_2, I_1)$.

(3) For all i_2, all sets W, and all $w_0 : W \to \Gamma(I_0, I_0)$ and $w_1 : W \to \Gamma(I_1, I_1)$ such that $\Gamma(I_0, i_2) \circ w_0 = \Gamma(i_2, I_1) \circ w_1$,

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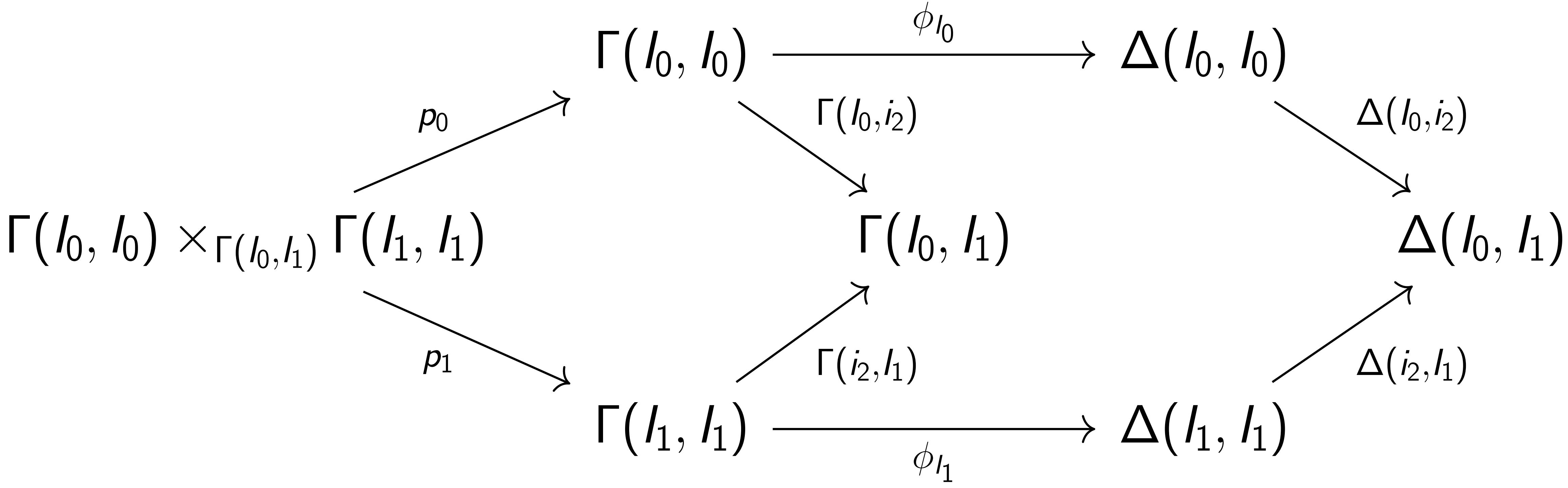
$\Gamma(I_0, I_0) \times_{\Gamma(I_0,I_1)} \Gamma(I_1, I_1)$
p_0
p_1
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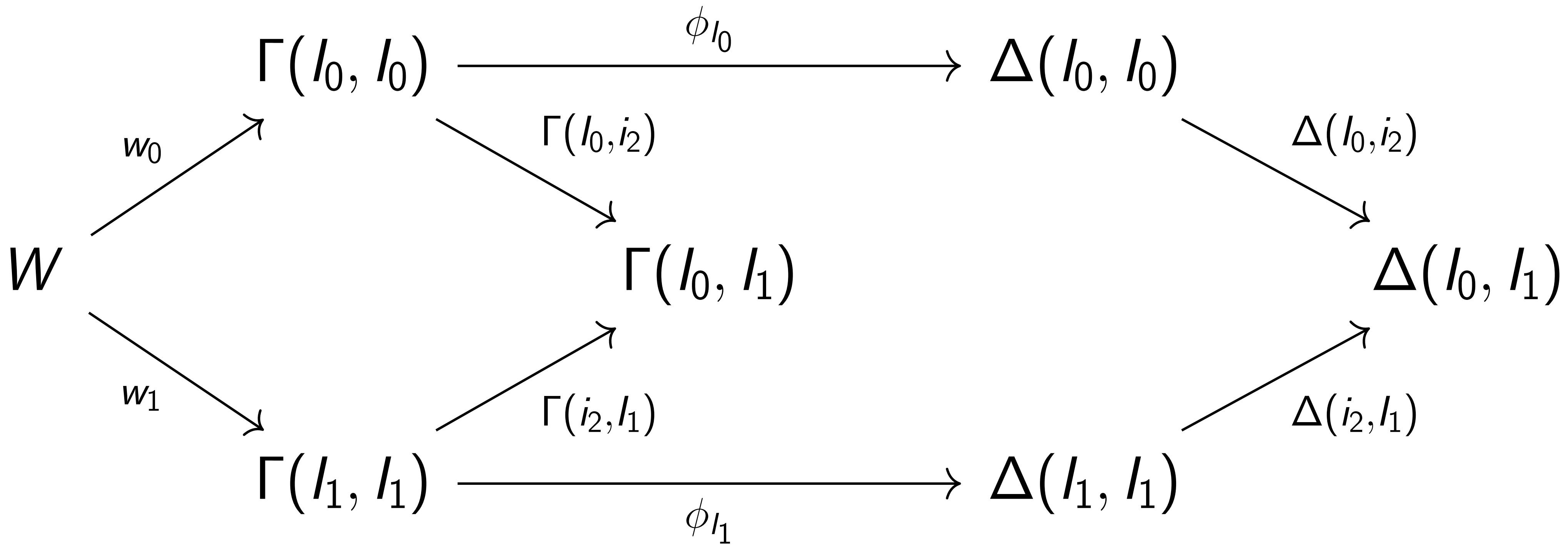

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$$\Delta(I_0, i_2) \circ \phi_{I_0} \circ w_0 = \Delta(i_2, I_1) \circ \phi_{I_1} \circ w_1$$

Structures of a difunctor

Defn. For any $\Gamma\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$, define the **category of Γ-structures** (or **category of diagonal elements of Γ**)—denoted Γ-Struct—to be the category

- whose objects are pairs
$$(I, g)\colon \sum_{I:|\mathbb{C}|} \Gamma(I, I)$$
- whose morphisms (I_0, g_0) to (I_1, g_1) are $\mathbb{C}$-morphisms $i_2\colon \mathrm{Hom}(I_0, I_1)$ such that
$$\Gamma(I_0, i_2)\ g_0 \quad = \quad \Gamma(i_2, I_1)\ g_1$$
(call these "Γ homomorphisms")

with identities and composition inherited from $\mathbb{C}$.

Observations about structs and paranaturality

Notice The paranaturality condition (for $\phi\colon \Gamma \overset{\diamond}{\to} \Delta$) can be rephrased as: *if* i_2 is a Γ homomorphism from (I_0, g_0) to (I_1, g_1), then i_2 is a Δ homomorphism from $(I_0, \phi_{I_0}\ g_0)$ to $(I_1, \phi_{I_1}\ g_1)$.

Notation If $\phi\colon \Gamma \overset{\diamond}{\to} \Delta$, write $\underline{\phi}$ for the functor Γ-Struct $\to$ Δ-Struct sending (I, g) to $(I, \phi_I\ g)$ and sending morphisms i_2 to themselves (which is functorial, by the above comment).

Claim The "underlining" operation (taking the corresponding functor) is *itself* functorial: $\underline{\psi \circ \phi} = \underline{\psi} \circ \underline{\phi}$.

Notation Write $\mathring{\mathbb{C}}$ for the category whose objects are difunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$ and whose morphisms are paranatural transformations.

Defn. The **diYoneda embedding** $\mathbf{yy} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathring{\mathbb{C}}$ is the functor whose object part is given by

$$\mathbf{yy}\ (I_0, I_1)\ (J_0, J_1) :\equiv \mathrm{Hom}(I_0, J_1) \times \mathrm{Hom}(J_0, I_1)$$

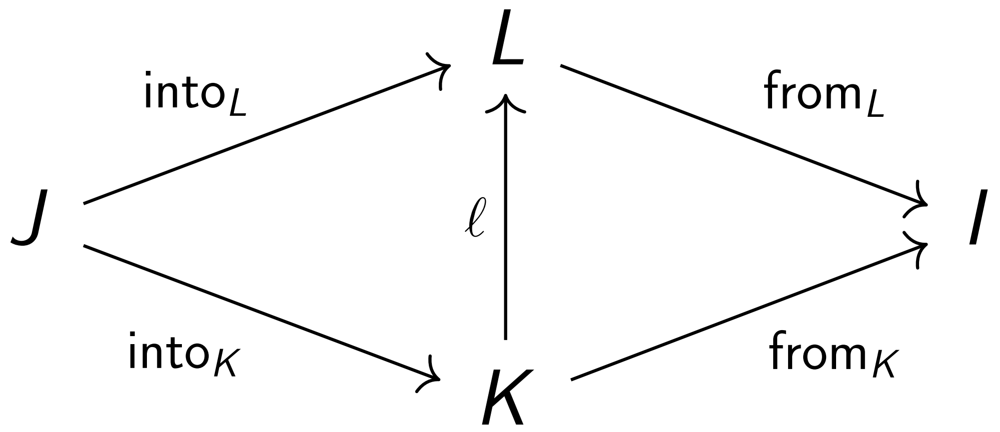
and whose four morphism parts are given by appropriate pre- and post-compositions.

Question: What is $\mathbf{yy}(J, I)$-Struct?

Defn. For objects $I, J \colon |\mathbb{C}|$, define the **splice category** $J/\mathbb{C}/I$ between J and I to be the category whose objects are diagrams of the form

$$J \xrightarrow{\mathrm{into}_K} K \xrightarrow{\mathrm{from}_K} I$$

and whose morphisms from $(K, \mathrm{into}_K, \mathrm{from}_K)$ to $(L, \mathrm{into}_L, \mathrm{from}_L)$ are maps $\ell \colon \mathrm{Hom}(K, L)$ making both triangles commute:



Lemma For any difunctor $\Delta\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Set}$, there is a bijection

$$\Delta(I, J) \quad \cong \quad \mathbf{yy}(J, I) \xrightarrow{\diamond} \Delta$$

paranatural in I, J.

- Note that I and J are flipped on the right
- A paranatural transformation is an iso iff its corresponding functor is
- To prove this, we construct an $\alpha_d\colon \mathbf{yy}(I, I) \xrightarrow{\diamond} \Delta$ for each $d\colon \Delta(I, I)$ and vice-versa.

Claim For any $\mathbb{C}$, the category of difunctors $\mathring{\mathbb{C}}$ has a terminal object (the constant-singleton difunctor) and binary products (given pointwise).

Prop. For any $\mathbb{C}$, the category of difunctors $\mathring{\mathbb{C}}$ has exponential objects.

Proof. By "diYoneda reasoning": for difunctors Γ, Δ, *suppose* their exponential Δ^Γ existed. Then

$$\Delta^\Gamma(I, J) \cong \mathbf{yy}(J, I) \overset{\diamond}{\to} \Delta^\Gamma \qquad \text{diYoneda Lemma}$$

$$\cong \mathbf{yy}(J, I) \times \Gamma \overset{\diamond}{\to} \Delta \qquad \text{(desired property)}$$

so now *define* $\Delta^\Gamma(I, J)$ to be $\mathbf{yy}(J, I) \times \Gamma \overset{\diamond}{\to} \Delta$, and verify this satisfies all the necessary properties.

Conjecture: $\overset{\diamond}{\mathbb{C}}$ is a (co)complete topos with pointwise colimits

3 Difunctor Models of Type Theory

Idea: Define models of type theory using difunctors

We'll be using CwFs as our notion of a "model of type theory". So we need:

- A category Con of **contexts** and **substitutions** (that has a terminal object, the empty context)
- A presheaf $\mathrm{Ty}\colon \mathrm{Con}^{\mathrm{op}} \to \mathrm{Set}$ of **types**
- A presheaf $\mathrm{Tm}\colon (\int \mathrm{Ty})^{\mathrm{op}} \to \mathrm{Set}$ of **terms**
- An operation of **context extension** (for $\Delta\colon \mathrm{Con}$ and $A\colon \mathrm{Ty}\,\Delta$, a specified $\Delta.A\colon \mathrm{Con}$) satisfying the appropriate condition.

Note about size and identity

So far, we haven't been paying attention to size issues or homotopy level (e.g. whether $\Gamma \xrightarrow{\diamond} \Delta$ constitutes a set). For this section, we'll be

- Assuming a Grothendieck set universe $\mathcal{U}$, whose elements we'll call "small sets".
- Assuming UIP (someone will need to go through and do a higher-homotopy version of this someday)

Difunctor models

Let $\mathbb{C}$ be a small category ($|\mathbb{C}|$ is in $\mathcal{U}$, as are all hom-sets). Then the **difunctor model of type theory** (on $\mathbb{C}$) is defined as follows.

- Con will be the the category $\mathring{\mathbb{C}}$ of difunctors and paranatural transforms on $\mathbb{C}$. The empty context, ♦, is the constant-singleton difunctor.
- A type in context Δ will be a small-set-valued difunctor on Δ-Struct, i.e. some $A \colon (\Delta\text{-Struct})^{\mathrm{op}} \times \Delta\text{-Struct} \to \mathcal{U}$. Type substitution (the morphism part of Ty) is defined by

$$A[\delta]\,(I, g)\,(I', g') :\equiv A\,(I, \delta_I\, g)\,(I', \delta_{I'}\, g')$$

for some $\delta \colon \Gamma \overset{\diamond}{\to} \Delta$.

- Given $A\colon \mathrm{Ty}\,\Delta$, a term $a\colon \mathrm{Tm}(\Delta, A)$ is a **dependent paranatural transformation** from Δ to A. That is, a dependent function

$$a_I \quad : \quad \prod_{d\colon \Delta(I,I)} A\,(I,d)\,(I,d)$$

for each $I\colon\ |\mathbb{C}|$, satisfying a "dependent paranaturality condition": if $\Delta(I_0, i_2)\ d_0 = \Delta(i_2, I_1)\ d_1$, then

$$A((I_0, d_0), i_2)\ (a_{I_0}\ d_0) = A(i_2, (I_1, d_1))\ (a_{I_1}\ d_1).$$

Difunctor models (cont.)

- Given $\Delta : \mathsf{Con}$ and $A : \mathsf{Ty}\,\Delta$, we want $\Delta.A$ to satisfy

$$\Gamma \xrightarrow{\diamond} \Delta.A \quad \cong \quad \sum_{\delta : \Gamma \xrightarrow{\diamond} \Delta} \mathsf{Tm}(\Gamma, A[\delta]).$$

So use diYoneda reasoning!

$$\begin{aligned} \Delta.A(I, J) &\cong \mathbf{yy}(J, I) \xrightarrow{\diamond} \Delta.A \\ &\cong \sum_{\delta : \mathbf{yy}(J,I) \xrightarrow{\diamond} \Delta} \mathsf{Tm}(\mathbf{yy}(J, I), A[\delta]) \end{aligned}$$

Can we simplify more? Open question: dependent diYoneda?

Using the difunctor/paranatural analogue of a trick due to Hofmann and Streicher, we can internalize our Grothendieck universe as a "large closed type" $\mathbf{U} \colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$ (note that ♦-Struct is isomorphic to $\mathbb{C}$ itself). Again, use diYoneda: we want $\mathbf{U}$ to satisfy

$$\mathsf{Tm}(\Gamma, \mathbf{U}) \quad \cong \quad \mathsf{Ty}\,\Gamma$$

so we have:

$$\begin{aligned}
\mathbf{U}(I, J) &\cong \mathbf{yy}(J, I) \xrightarrow{\diamond} \mathbf{U} \\
&\equiv \mathsf{Tm}(\mathbf{yy}(J, I), \mathbf{U}) \\
&\cong \mathsf{Ty}(\mathbf{yy}(J, I)) \\
&\equiv (\mathbf{yy}(J, I)\text{-}\mathsf{Struct})^{\mathsf{op}} \times \mathbf{yy}(J, I)\text{-}\mathsf{Struct} \to \mathcal{U} \\
&\equiv (J/\mathbb{C}/I)^{\mathsf{op}} \times (J/\mathbb{C}/I) \to \mathcal{U}.
\end{aligned}$$

Future directions

- Dependent diYoneda Lemma
- Difunctor semantics of HOAS/SOGATs
- Internal parametricity
- Directed type theory connection