DTL, Refined

Topology, Knowledge, and Nondeterminism

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References/Current versions

- *Epistemology of Nondeterminism* (Bjorndahl, 2018)
- *Semantics of Nondeterministic Construction* (N., 2020)
- Current "math draft": `jacobneu.github.io/research/char.pdf`
- These slides: `jacobneu.github.io/research/pfew-2021.pdf`

Big Picture:

Dynamic Topological Logic provides a useful language to articulate & model dynamic-epistemic situations

Methodology

1. Interpret the formal symbols of DTL as making statements about dynamic-epistemic situations
2. Encode the features of dynamic-epistemic situations in the mathematical theory of DTL
3. Mathematically analyze the resulting models and obtain axiomatizations of them
4. Check the axioms against our interpretation

My Goal:

Convince you that this strategy works (at least in the case of coin-flipping situations)

Outline

1 Nondeterministic Union

A coin-flipping scenario

Alice walks into a coffee shop, but can't decide whether to get a latte or an americano. So she pulls a quarter out of her pocket and says, "if I flip this coin and it comes up heads, I'll get a latte. If it comes up tails, I'll get an americano." She flips it, it comes up tails, and she orders an americano

A Philosophical Question:

What *exactly* is going on here? How does Alice's knowledge evolve through this scenario?

The Philosophy of Coin-Flipping

- Many familiar accounts of coin-flipping focus on the agent's degrees of belief about how the coin will come up
- Our question is more basic: what can the agent *know*?
  - Alice knows before the flip that she'll definitely be drinking a drink containing espresso
  - Alice *doesn't* know whether she'll be drinking a drink containing milk
- Only when we've established that the agent cannot know the outcome of the flip does it become interesting to consider the agent's degrees of belief
- The mathematical structures we'll use are quite amenable to the attachment of probability theory

2 $\mathcal{L}^{or}_{\square\bigcirc}$ Syntax

Complexities we're ignoring

- The flip could be interrupted or inconclusive
- The agent could "disobey" the result
- The action of flipping could change the state of the world or the available actions

Nondeterministic union as a binary operation on actions

We assume that the "basic" or "primitive" actions are all denoted by some element of the set Π. We then close Π under the binary function symbol 'or':

$$\pi_0 \text{ or } \pi_1 \quad : \quad \textit{flip a coin. If heads, do } \pi_0\textit{; if tails, do } \pi_1$$

Π^{or} denotes the least set containing Π which is closed under or:

$$\sigma ::= \pi \mid \sigma_0 \text{ or } \sigma_1 \qquad (\pi \in \Pi)$$

Formal Language for the Consequences of Actions

For any set Σ of actions (such as Π or Π^{or}), $\mathcal{L}_{\bigcirc}(\Sigma)$ is the language given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \land \psi \mid \bigcirc_\sigma \varphi \qquad (\sigma \in \Sigma)$$

$\bigcirc_\sigma \varphi$	:	*after performing σ (from the present world), φ will be the case*
	:	*i.e. performing σ will take us to a φ world*

Goal:

Understand the formal relationship between $\bigcirc_{\pi_0}$, $\bigcirc_{\pi_1}$ and $\bigcirc_{\pi_0 \text{or} \pi_1}$

One complexity we're allowing: non-totality

The action σ could perhaps be impossible from the present world, i.e. σ *lacks extension*

We choose the convention that $\bigcirc_\sigma \varphi$ is vacuously *false* if σ lacks extension at the present world. Alternate conventions are possible.

Special “Actions”

We have two special action names, which we regard as implicitly part of Π or Π^{or}, and whose semantics will be fixed in the theory:

- skip is the “do-nothing” action, which doesn't change the world ($\models \varphi \leftrightarrow \bigcirc_{\text{skip}}\varphi$)
- abort is the “nowhere-defined” action, which lacks extension everywhere ($\models \neg \bigcirc_{\text{abort}} \top$)

Add in knowability

$\mathcal{L}_{\Box\bigcirc}(\Sigma)$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \bigcirc_\sigma \varphi \mid \Box\varphi \qquad (\sigma \in \Sigma)$$

$\Box\varphi$	:	*φ is knowably true in the present world*
	:	*the agent could come to know φ*
	:	*there exists some evidence which would allow the agent to conclude φ*

Diamond

Formally, $\Diamond$ is defined as an abbreviation for $\neg\Box\neg$.

$\Diamond\varphi$	:	*as far as the agent could know, φ could be true*
	:	*the agent's knowledge-gathering abilities do not allow her to rule out φ*
	:	*either φ is true, or else the agent cannot know that φ is false*

Diamond Circle

$\Diamond\bigcirc_\sigma\varphi$ says something like "as far as the agent can know, executing σ could result in a φ world". Note that this is not incompatible with $\Diamond\bigcirc_\sigma\neg\varphi$.

We might abbreviate $\Diamond\bigcirc_\sigma\varphi$ as $\langle\sigma\rangle\varphi$, to identify it with the "diamond-style" modality in classical PDL. This connection will be vindicated by the semantics, and shows off our more sophisticated notion of nondeterminism.

3 The DTL + $\chi_{\mathrm{OR}\{\omega\}}$ Axioms

DTL Axioms

S4 for Knowability

- $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$
- $\Box\varphi \rightarrow \varphi$
- $\Box\varphi \rightarrow \Box\Box\varphi$
- from φ, deduce $\Box\varphi$

Axioms for $\bigcirc$

- $\bigcirc_\sigma \neg\varphi \leftrightarrow (\neg\bigcirc_\sigma\varphi \wedge \bigcirc_\sigma\top)$
- $\bigcirc_\sigma(\varphi \wedge \psi) \leftrightarrow \bigcirc_\sigma\varphi \wedge \bigcirc_\sigma\psi$
- from $\varphi \rightarrow \psi$, deduce $\bigcirc_\sigma\varphi \rightarrow \bigcirc_\sigma\psi$

χOR{ω}

Maybe

$$\mathsf{Maybe}_0(\varphi, \pi_0, \pi_1) \quad :\equiv \quad \bigcirc_{\pi_0}\varphi \leftrightarrow \bigcirc_{(\pi_0 \text{ or } \pi_1)}\varphi$$

$$\mathsf{Maybe}_1(\varphi, \pi_0, \pi_1) \quad :\equiv \quad \bigcirc_{\pi_1}\varphi \leftrightarrow \bigcirc_{(\pi_0 \text{ or } \pi_1)}\varphi$$

OR Typicality

For all atomic props (or $\top$, $\bot$) p, and all $\pi_0, \pi_1 \in \Pi^{\text{or}}$ $(\cup \{\text{skip}, \text{abort}\})$

$$\text{Maybe}_0(p, \pi_0, \pi_1) \ \vee \ \text{Maybe}_1(p, \pi_0, \pi_1)$$

Either the coin comes up heads or it comes up tails

Only

- $\text{Only}_0(\varphi, \pi_0, \pi_1) :\equiv$

  $(\bigcirc_{\pi_0}\varphi \wedge \neg \bigcirc_{\pi_1} \varphi \wedge \bigcirc_{\pi_0 \text{ or } \pi_1}\varphi) \vee (\neg \bigcirc_{\pi_0} \varphi \wedge \bigcirc_{\pi_1}\varphi \wedge \neg \bigcirc_{\pi_0 \text{ or } \pi_1} \varphi)$

- $\text{Only}_1(\varphi, \pi_0, \pi_1) :\equiv$

  $(\neg \bigcirc_{\pi_0} \varphi \wedge \bigcirc_{\pi_1}\varphi \wedge \bigcirc_{\pi_0 \text{ or } \pi_1}\varphi) \vee (\bigcirc_{\pi_0}\varphi \wedge \neg \bigcirc_{\pi_1} \varphi \wedge \neg \bigcirc_{\pi_0 \text{ or } \pi_1} \varphi)$

OR Realization

$$p \wedge \mathrm{Only}_1(q, \pi_0, \pi_1) \rightarrow \Diamond(p \wedge \mathrm{Only}_0(q, \pi_0, \pi_1))$$
$$p \wedge \mathrm{Only}_0(q, \pi_0, \pi_1) \rightarrow \Diamond(p \wedge \mathrm{Only}_1(q, \pi_0, \pi_1))$$

Both outcomes of the flip are possible (as far as the agent can know)

OR Regularity

$$\text{Only}_0(p, \pi_0, \pi_1) \rightarrow \text{Maybe}_0(q, \pi_2, \pi_3)$$
$$\text{Only}_1(p, \pi_0, \pi_1) \rightarrow \text{Maybe}_1(q, \pi_2, \pi_3)$$

The result of the coin flip is independent of *what you're using it to decide*

Other Axioms

- OR Refresh:

$$p \rightarrow \bigcirc_{(\text{skip or skip})} p$$

- OR Nesting:

$$\text{Only}_0(p, \pi_0, \pi_1) \rightarrow \bigcirc_{(\text{skip or skip})} \bigcirc_{\sigma_0} \varphi \rightarrow \bigcirc_{(\sigma_0 \text{ or } \sigma_1)} \varphi$$
$$\text{Only}_1(p, \pi_0, \pi_1) \rightarrow \bigcirc_{(\text{skip or skip})} \bigcirc_{\sigma_1} \varphi \rightarrow \bigcirc_{(\sigma_0 \text{ or } \sigma_1)} \varphi$$

- OR Primitive Independence:

$$\bigcirc_{(\text{skip or skip})} \bigcirc_{\pi_0} \varphi \leftrightarrow \bigcirc_{\pi_0} \bigcirc_{(\text{skip or skip})} \varphi$$
$$\text{Only}_0(\top, \text{skip}, \text{abort}) \rightarrow \neg \bigcirc_{\pi_0} \neg \text{Only}_0(\top, \text{skip}, \text{abort})$$
$$\text{Only}_1(\top, \text{skip}, \text{abort}) \rightarrow \neg \bigcirc_{\pi_0} \neg \text{Only}_1(\top, \text{skip}, \text{abort})$$

Classical PDL Axiom

PDL has an axiom governing how or should work, the (U) axiom:

$$\Diamond\bigcirc_{(\sigma_0 \text{ or } \sigma_1)}\varphi \leftrightarrow \Diamond\bigcirc_{\sigma_0}\varphi \vee \langle\sigma_1\rangle\Diamond\bigcirc_{\sigma_1}\varphi$$

Conj. $\chi_{\mathsf{OR}\{\omega\}} \vdash_{DTL} (\mathsf{U})$

4 Semantics & Methodology

Dynamic Topological Models

Our basic structure is that of a Σ-DTM:

Defn. Given a set Σ of actions, a **Σ-DTM** is a 4-tuple

$$\mathfrak{M} = \left(|\mathfrak{M}|\,, \tau_{\mathfrak{M}}, \{\|\sigma\|_{\mathfrak{M}}\}_{\sigma\in\Sigma}\,, V_{\mathfrak{M}}\right)$$

where

- $|\mathfrak{M}|$ is some (nonempty) set
- $\tau_{\mathfrak{M}}$ is a topology on $|\mathfrak{M}|$
- $\|\sigma\|_{\mathfrak{M}} : |\mathfrak{M}| \rightharpoonup |\mathfrak{M}|$ for each $\sigma \in \Sigma$
- $V_{\mathfrak{M}}$ sends atomic propositions p to their extension $V_{\mathfrak{M}}(p) \subseteq |\mathfrak{M}|$

Semantics

Σ-DTMs give semantics for $\mathcal{L}_{\Box\bigcirc}(\Sigma)$: for each Σ-DTM $\mathfrak{M}$,

$$[\![-]\!]_{\mathfrak{M}} : \mathcal{L}_{\Box\bigcirc}(\Sigma) \to \mathcal{P}(|\mathfrak{M}|)$$

$$\begin{aligned} [\![p]\!]_{\mathfrak{M}} &= V_{\mathfrak{M}}(p) \\ [\![\neg\varphi]\!]_{\mathfrak{M}} &= |\mathfrak{M}| \setminus [\![\varphi]\!]_{\mathfrak{M}} \\ [\![\varphi \wedge \psi]\!]_{\mathfrak{M}} &= [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}} \end{aligned}$$

Knowledge is interior

$$[\![\Box\varphi]\!]_{\mathfrak{M}} \quad = \quad \text{int}([\![\varphi]\!]_{\mathfrak{M}})$$

int denotes topological interior (with respect to $\tau_{\mathfrak{M}}$):

$$\begin{aligned} x \in \text{int}([\![\varphi]\!]) &\iff \exists U \in \tau_{\mathfrak{M}} \text{ s.t. } x \in U \subseteq [\![\varphi]\!] \\ &\iff \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq [\![\varphi]\!] \end{aligned}$$

where $\mathcal{B}$ is a basis for the topology $\tau_{\mathfrak{M}}$.

Partial Functions interpret actions

$$x \in [\![\bigcirc_\sigma \varphi]\!]_{\mathfrak{M}} \quad \Longleftrightarrow \quad \|\sigma\|_{\mathfrak{M}}(x) \text{ is defined and } \|\sigma\|_{\mathfrak{M}}(x) \in [\![\varphi]\!]_{\mathfrak{M}}$$

- The extension of each action σ is a transformation sending worlds to worlds
- $\bigcirc_\sigma \varphi$ is true iff the extension of σ will take you to a φ world
- $\|\sigma\|_{\mathfrak{M}}$ can be undefined at x, making $\bigcirc_\sigma \varphi$ false at x
- Stipulate: $\|\text{skip}\|$ is the identity function, $\|\text{abort}\| = \emptyset$

Semantics

Σ-DTMs give semantics for $\mathcal{L}_{\Box\bigcirc}(\Sigma)$: for each Σ-DTM $\mathfrak{M}$,

$$[\![-]\!]_{\mathfrak{M}} : \mathcal{L}_{\Box\bigcirc}(\Sigma) \to \mathcal{P}(|\mathfrak{M}|)$$

$$\begin{aligned}
[\![p]\!]_{\mathfrak{M}} &= V_{\mathfrak{M}}(p) \\
[\![\neg\varphi]\!]_{\mathfrak{M}} &= |\mathfrak{M}| \setminus [\![\varphi]\!]_{\mathfrak{M}} \\
[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} &= [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}} \\
[\![\bigcirc_\sigma]\!]_{\mathfrak{M}} &= \|\sigma\|_{\mathfrak{M}}^{-1}([\![\varphi]\!]_{\mathfrak{M}}) \\
[\![\Box\varphi]\!]_{\mathfrak{M}} &= \mathrm{int}([\![\varphi]\!]_{\mathfrak{M}})
\end{aligned}$$

Write $(\mathfrak{M}, x) \models \varphi$ if $x \in [\![\varphi]\!]_{\mathfrak{M}}$, and $\mathfrak{M} \models \varphi$ if $[\![\varphi]\!]_{\mathfrak{M}} = |\mathfrak{M}|$.

Frames

Defn. A Σ-frame is a triple

$$\mathcal{F} = (|\mathcal{F}|, \tau_{\mathcal{F}}, \{\|\sigma\|_{\mathcal{F}}\}_{\sigma \in \Sigma})$$

A Σ-DTM $\mathfrak{M}$ is **based on** $\mathcal{F}$ if $|\mathfrak{M}| = |\mathcal{F}|$, $\tau_{\mathfrak{M}} = \tau_{\mathcal{F}}$, and $\|\sigma\|_{\mathfrak{M}} = \|\sigma\|_{\mathcal{F}}$ for all σ.

$$\begin{aligned} \mathcal{F} \models \varphi \quad & \iff \quad \mathfrak{M} \models \varphi \text{ for all } \mathfrak{M} \text{ based on } \mathcal{F} \\ & \iff \quad (\mathcal{F}, V) \models \varphi \text{ for all } V \end{aligned}$$

Augmentation

We want to study Π^{or}-DTMs/frames, but we want $\|\sigma_0 \text{ or } \sigma_1\|$ to combine $\|\sigma_0\|$ and $\|\sigma_1\|$ in the proper way.
View this as an "augmentation" process:

$$\begin{array}{ccc} \mathfrak{M} \text{ a } \Pi\text{-DTM} & \rightsquigarrow & \mathfrak{M}^{\mathrm{OR}} \text{ a } \Pi^{\mathrm{or}}\text{-DTM} \\ \mathcal{F} \text{ a } \Pi\text{-frame} & \rightsquigarrow & \mathcal{F}^{\mathrm{OR}} \text{ a } \Pi^{\mathrm{or}}\text{-frame} \end{array}$$

In $\mathfrak{M}^{\mathrm{OR}}$ and $\mathcal{F}^{\mathrm{OR}}$, the or-actions σ_0 or σ_1 will be interpreted in a way which matches our intuitions about coin-flipping

Goldilocks

Goal: want to obtain a set Δ of $\mathcal{L}_{\Box\bigcirc}(\Pi^{\text{or}})$ formulas such that

$$\mathcal{G} \models \Delta \quad \Longleftrightarrow \quad \mathcal{G} \simeq \mathcal{F}^{\text{OR}} \text{ for some } \Pi\text{-frame } \mathfrak{M}\mathcal{F}$$

But...

- Model-level satisfaction is too specific, so this proves impossible (Thm from thesis)
- Frame-level satisfaction is too broad, and so also does not work for our purposes

Refined Frames

We invent an intermediate notion, *refined frames*:

Defn. A **refined frame** is a pair $(\mathcal{G}, \mathcal{R})$ where $\mathcal{G}$ is a Π^{or}-frame and $\mathcal{R}$ is an equivalence relation on $|\mathcal{G}|$ satisfying certain conditions.

Defn. A valuation V on $\mathcal{G}$ is said to **respect** $\mathcal{R}$ if

$$x\mathcal{R}x' \implies (x \in V(p) \iff x' \in V(p) \text{ for all } p)$$

$$(\mathcal{G}, \mathcal{R}) \models \varphi \iff (\mathcal{G}, V) \models \varphi \text{ for all } V \text{ which respect } \mathcal{R}$$

Advantages

- Intermediate to model- and frame-level satisfaction: we can use $\mathcal{R}$ to specify which valuations to consider
- $\mathcal{R}$ encodes which worlds are supposed to be "copies" of each other

Idea: $\mathcal{R}$-related worlds are "Π-indistinguishable", but may differ on how they interpret or-actions

Refinement and Indistinguishability

RR1 $\mathcal{R}$ is an equivalence relation

RR2 For all $\pi \in \Pi$, if $x\mathcal{R}x'$, then

$$\|\pi\|(x) \text{ is defined} \iff \|\pi\|(x') \text{ is defined}$$

and, if both are defined,

$$(\|\pi\|(x))\ \mathcal{R}\ (\|\pi\|(x'))$$

RR3 If $U \subseteq |\mathcal{G}|$ is any open set, then

$$\mathcal{R}(U) = \{w \in |\mathcal{G}| \ : \ w\mathcal{R}w' \text{ for some } w' \in U\}$$

is an open set

OR $\{\omega\}$

$$\mathcal{F} \text{ a } \Pi\text{-frame} \quad \rightsquigarrow \quad \left(\mathcal{F}^{\mathsf{OR}\{\omega\}}, \mathcal{R}_{\mathcal{F}}^{\mathsf{OR}\{\omega\}}\right) \text{ a refined frame}$$

- $\left|\mathcal{F}^{\mathsf{OR}\{\omega\}}\right| = |\mathcal{F}| \times \{0,1\}^{\omega}$
-

$$\|\pi\|_{\mathcal{F}^{\mathsf{OR}\{\omega\}}}(x,\gamma) = (\|\pi\|_{\mathcal{F}}(x),\gamma) \qquad (\pi \in \Pi)$$

$$\|\sigma_0 \text{ or } \sigma_1\|_{\mathcal{F}^{\mathsf{OR}\{\omega\}}}(x,\gamma) = \begin{cases} \|\sigma_0\|_{\mathcal{F}^{\mathsf{OR}\{\omega\}}}(x,\mathsf{tl}(\gamma)) & \text{if } \mathsf{hd}(\gamma) = 0 \\ \|\sigma_1\|_{\mathcal{F}^{\mathsf{OR}\{\omega\}}}(x,\mathsf{tl}(\gamma)) & \text{if } \mathsf{hd}(\gamma) = 1 \end{cases}$$

OR $\{\omega\}$

$$\mathcal{F} \text{ a } \Pi\text{-frame} \quad \rightsquigarrow \quad \left(\mathcal{F}^{\mathsf{OR}\{\omega\}}, \mathcal{R}_{\mathcal{F}}^{\mathsf{OR}\{\omega\}}\right) \text{ a refined frame}$$

- $\left|\mathcal{F}^{\mathsf{OR}\{\omega\}}\right| = |\mathcal{F}| \times \{0,1\}^{\omega}$
- Topology: product topology of $\tau_{\mathcal{F}}$ and the *indiscrete* topology on $\{0,1\}^{\omega}$
- $\mathcal{R}_{\mathcal{F}}^{\mathsf{OR}\{\omega\}}$ relates (x,γ) to (x,γ'), (x,γ''), …

Understanding OR $\{\omega\}$

- For each world x of $\mathcal{F}$, there are $\{0,1\}^\omega$ many "copies" (x,γ) of x in $\mathcal{F}^{\mathsf{OR}\{\omega\}}$
- These copies are all $\mathcal{R}_{\mathcal{F}}^{\mathsf{OR}\{\omega\}}$-related
- The different copies of x interpret $\pi \in \Pi$ the same way (ignoring the $\gamma \in \{0,1\}^\omega$)
- For a world (x,γ) of $\mathcal{F}^{\mathsf{OR}\{\omega\}}$, γ encodes the outcome of all future coin-flips, giving interpretation to or
- The agent cannot know anything about what γ is (no open set distinguishes (x,γ) from $(x,\gamma'))$

Reinterpreting $\chi_{\mathsf{OR}\{\omega\}}$

- Typicality:

$$\mathsf{Maybe}_0(p, \pi_0, \pi_1) \lor \mathsf{Maybe}_1(p, \pi_0, \pi_1)$$

- Realization:

$$p \land \mathsf{Only}_1(q, \pi_0, \pi_1) \to \Diamond(p \land \mathsf{Only}_0(q, \pi_0, \pi_1))$$
$$p \land \mathsf{Only}_0(q, \pi_0, \pi_1) \to \Diamond(p \land \mathsf{Only}_1(q, \pi_0, \pi_1))$$

- Regularity:

$$\mathsf{Only}_0(p, \pi_0, \pi_1) \to \mathsf{Maybe}_0(q, \pi_2, \pi_3)$$
$$\mathsf{Only}_1(p, \pi_0, \pi_1) \to \mathsf{Maybe}_1(q, \pi_2, \pi_3)$$

Reinterpreting $\chi_{\mathsf{OR}\{\omega\}}$

- OR Refresh:

$$p \to \bigcirc_{(\text{skip or skip})} p$$

- OR Nesting:

$$\text{Only}_0(p, \pi_0, \pi_1) \to \bigcirc_{(\text{skip or skip})} \bigcirc_{\sigma_0} \varphi \to \bigcirc_{(\sigma_0 \text{ or } \sigma_1)} \varphi$$

$$\text{Only}_1(p, \pi_0, \pi_1) \to \bigcirc_{(\text{skip or skip})} \bigcirc_{\sigma_1} \varphi \to \bigcirc_{(\sigma_0 \text{ or } \sigma_1)} \varphi$$

- OR Primitive Independence:

$$\bigcirc_{(\text{skip or skip})} \bigcirc_{\pi_0} \varphi \leftrightarrow \bigcirc_{\pi_0} \bigcirc_{(\text{skip or skip})} \varphi$$

$$\text{Only}_0(\top, \text{skip}, \text{abort}) \to \neg \bigcirc_{\pi_0} \neg \text{Only}_0(\top, \text{skip}, \text{abort})$$

$$\text{Only}_1(\top, \text{skip}, \text{abort}) \to \neg \bigcirc_{\pi_0} \neg \text{Only}_1(\top, \text{skip}, \text{abort})$$

Right-Characterization

- $\left(\mathcal{F}^{\mathsf{OR}\{\omega\}}, \mathcal{R}_{\mathcal{F}}^{\mathsf{OR}\{\omega\}}\right) \models \chi_{\mathsf{OR}\{\omega\}}$ for all Π-frames $\mathcal{F}$
- If $(\mathcal{G}, \mathcal{R})$ is a refined frame validating all of $\chi_{\mathsf{OR}\{\omega\}}$, then $(\mathcal{G}, \mathcal{R})$ can be isomorphically embedded in some refined frame of the form $\left(\mathcal{F}^{\mathsf{OR}\{\omega\}}, \mathcal{R}_{\mathcal{F}}^{\mathsf{OR}\{\omega\}}\right)$

5 Conclusion & Future Directions

Summary

1. Interpret the formal symbols of DTL as making statements about dynamic-epistemic situations (or as "coin-flipping")
2. Encode the features of dynamic-epistemic situations in the mathematical theory of DTL ($\mathsf{OR}\,\{\omega\}$)
3. Mathematically analyze the resulting models and obtain axiomatizations of them ($\chi_{\mathsf{OR}\{\omega\}}$)
4. Check the axioms against our interpretation

Other versions of OR

- OR (thesis)
- OR $\{1\}$, OR $\{2\}$, ...
- OR $\{<\omega\}$
- OR $\{\leq\omega\}$

Other directions

- Other program constructors (e.g. STAR, If/then/else, looping)
- Richer structure (probability, more elaborate valuations, knowledge, time, etc.)
- Non-arbitrary decisions (e.g. utility calculations, deontology, etc.)

Thank You!