Allegories and Bisimulations

A category-theoretic take on relations

CMU HoTT Graduate Workshop
Jacob Neumann
February 2021

Outline

- Section 0: Allegories

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- Section 1: Back-and-Forth

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- Section 1: Back-and-Forth
- Section 2: Modal Logic (time-permitting)

Some sources

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- https://ncatlab.org/nlab/show/allegory
- Wikipedia has a good article: https://en.wikipedia.org/wiki/Allegory_(mathematics)

0 Background: the Allegory of Relations

Rel

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- composition operation is given by

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \ (a, b) \in R \ \& \ (b, c) \in S\}$$

for $R \in \mathrm{hom}_{\mathbf{Rel}}(A, B), S \in \mathrm{hom}_{\mathbf{Rel}}(B, C)$.

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- has cartesian closed slice categories
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- $\subseteq$ is compatible with composition: for $R : A \to B$, $S, S' : B \to C$ and $T : C \to D$ in **Rel**,

$$S \subseteq S' \implies (S \circ R) \subseteq (S' \circ R)$$

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\begin{aligned}
\hom_{\mathbf{Rel}}(-, C) &: \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Pos} \\
&: A \qquad\qquad\; \mapsto (\hom_{\mathbf{Rel}}(A, C), \subseteq) \\
&: R \subseteq A \times B \;\; \mapsto (- \circ R) : (\hom_{\mathbf{Rel}}(B, C), \subseteq) \to (\hom_{\mathbf{Rel}}(A, C), \subseteq) \\
\hom_{\mathbf{Rel}}(C, -) &: \mathbf{Rel} \to \mathbf{Pos} \\
&: D \qquad\qquad\; \mapsto (\hom_{\mathbf{Rel}}(C, D), \subseteq) \\
&: U \subseteq D \times E \;\; \mapsto (U \circ -) : (\hom_{\mathbf{Rel}}(C, D), \subseteq) \to (\hom_{\mathbf{Rel}}(C, E), \subseteq)
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- $R \cap R = R, \quad R \cap R' = R' \cap R, \quad R \cap (R' \cap R'') = (R \cap R') \cap R''$

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Also: nullary intersections ($A \times B \in \hom_{\mathbf{Rel}}(A, B)$), infinitary intersections, binary and infinitary unions, nullary unions (the empty relation), etc.

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- $\dagger$ is an **involution**: $\dagger \circ \dagger = \mathrm{id}_{\mathbb{C}}$

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- $f^\dagger$ is only a function if f is a bijection

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- The **modular law** is satisfied:

$$(S \circ R) \wedge T \leq (S \wedge (T \circ R^\dagger)) \circ R$$

Rel is an allegory

For any $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq A \times C$:

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$$\begin{aligned}&(b,c) \in S \text{ and } (b,c) \in T \circ R^\dagger \text{ and } (a,b) \in R \\ \iff\ &(b,c) \in S \cap (T \circ R^\dagger) \text{ and } (a,b) \in R \\ \implies\ &(a,c) \in (S \cap (T \circ R^\dagger)) \circ R\end{aligned}$$

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Note: if $R \in \hom_{\mathbb{C}}(A, B)$ for some allegory $\mathbb{C}$,

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Note **Set** is the subcategory of simple, entire relations

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Prop. R is entire iff $R^\dagger$ is coentire (and similarly for (co)simple)

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Prop. **Set** is a regular category, and $\mathbf{Rel} = \mathbf{Rel}(\mathbf{Set})$

1 Allegories with Back-and-Forth Classes

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Defn. **Top** is the category whose

- objects are topological spaces: pairs (X, τ) where τ is a topology on X
- morphisms are **continuous functions**:

Top

Defn. A **topology** on a set X is a collection $\tau \subseteq \mathcal{P}(X)$ (the elements of τ are called **open subsets** of X) such that

- $\emptyset, X \in \tau$
- If $U, U' \in \tau$, then $U \cap U' \in \tau$
- If I is a set and $U_i \in \tau$ for each $i \in I$,

$$\left(\bigcup_{i \in I} U_i\right) \in \tau.$$

Defn. **Top** is the category whose

- objects are topological spaces: pairs (X, τ) where τ is a topology on X
- morphisms are **continuous functions**: $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous if

$$U \in \tau_Y \implies f^{-1}(U) \in \tau_X$$

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Problem: R continuous does not imply $R^{\dagger}$ continuous, so the category of continuous relations (which *is* a category), is *not* an allegory.

Proposed approach: **TopRel**

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Prop. **TopRel** is an allegory.

Proof is identical to the proof that **Rel** is an allegory

Back and Forth

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- Quotient maps $X \to X/\sim$ in **Top** are open and coentire, but are only cosimple if $\sim$ is identity
- A **TopRel**-iso (a bijection) is a **Top**-iso (a homeomorphism) iff it is in **Forth** and **Back**

Another example: **DynRel**

Defn. A **dynamic set** is a pair (A, f) where A is a set and $f : A \rightharpoonup A$ is a partial function (a simple **Rel**-endomorphism).

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Parametrized: Σ-**DynRel**

Defn. For a set Σ, a **Σ-dynamic set** is a set A equipped with a Σ-indexed family of partial functions $\{f_\sigma : A \rightharpoonup A\}_{\sigma \in \Sigma}$.

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So, for each binary relation R, there is some subset $\Pi \subseteq \Sigma$ of all those σ such that R is in σ-**Forth** (or σ-**Back**, or both).

2 Modal Logics and Bisimulation

Valuations

The model theory of *classical logic* makes use of **valuations**: functions which "assign truth values" to atomic propositions

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We can generalize this somewhat:

$$v : \Phi \to \mathcal{P}(X)$$

$v(p)$ is the *extension* of p, or the set of "states where p is true".

Dynamic Modal Logic

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We define a function $[\![-]\!] : \mathcal{L}_{\bigcirc} \to \mathcal{P}(A)$ recursively by

$$\begin{aligned} [\![p]\!] &= v(p) && (p \in \Phi) \\ [\![\neg\varphi]\!] &= A \setminus [\![\varphi]\!] \\ [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\bigcirc\varphi]\!] &= f^{-1}[\![\varphi]\!] \end{aligned}$$

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Bisimulations of Dynamic Models

Defn. A **bisimulation** between dynamic models (A, f, v_A) and (B, g, v_B) is a binary relation $S \in \hom_{\mathbf{DynRel}}((A, f), (B, g))$ in both the **Forth** and **Back** classes, which also satisfies the **Base** condition: for any $(a, b) \in S$ and $p \in \Phi$,

$$a \in v_A(p) \qquad \Longleftrightarrow \qquad b \in v_B(p).$$

Bisimulation Invariance

Thm. For dynamic models (A, f, v_A) and (B, g, v_B) and a bisimulation S between them,

- If $(a, b) \in S$, then for any $\varphi \in \mathcal{L}_{\bigcirc}$,

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For $\Pi \subseteq \Sigma$, a Π-bisimulation between Σ-dynamic models $(A, \{f_\sigma\}_{\sigma\in\Sigma}, v_A)$ and $(B, \{g_\sigma\}_{\sigma\in\Sigma}, v_B)$ is a relation satisfying **Base**, and π-**Forth** and π-**Back** for each $\pi \in \Pi$.

Topological Modal Logic

We can instead use a topological structure to interpret $\Box$. Define $\mathcal{L}_\Box$ by

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A **topological model** (A, τ_A, v) interprets $\mathcal{L}_\Box$:

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where int denotes topological interior (with respect to τ_A).

Topo-bisimulations

Defn. A **topo-bisimulation** between topological models (A, τ_A, v_A) and (B, τ_B, v_B) is a **TopRel**-morphism in **Forth** and **Back** (open & continuous) that satisfies **Base**.

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Dynamic Topological Modal Logic

My research (in particular my master's thesis) explores bisimulations of **dynamic topological models**, which are models $(A, \tau_A, \{f_\sigma\}_{\sigma \in \Sigma}, v_A)$ interpreting

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with the appropriate notion of bisimulation.

This has an interesting philosophical interpretation if we read $\Box\varphi$ as "φ is knowably (or verifiably) true" and $\bigcirc_\sigma\varphi$ as "after performing (or executing) σ, φ holds".

Thank you!