Updates on Paranatural Category Theory

Jacob Neumann

University of Nottingham, United Kingdom
jacob.neumann@nottingham.ac.uk

It has been long-recognized[1] in the theory of functional programming languages that the concept of *parametric polymorphism* has a particular connection to the category-theoretic notion of *naturality*. For instance, Wadler [Wad89] famously noted that any System F function

$$r \quad : \quad \forall \alpha.\, \mathsf{List}(\alpha) \to \mathsf{List}(\alpha),$$

that is, a function $r_\alpha \colon \mathsf{List}(\alpha) \to \mathsf{List}(\alpha)$ which is "*polymorphic* in α", must automatically be a *natural transformation* from the List functor to itself, i.e.

$$r_Y \circ (\mathsf{map}\ f) = (\mathsf{map}\ f) \circ r_X$$

for any function $f \colon X \to Y$. However, this tight connection between parametricity and naturality breaks down for types with more complex variance; for instance, a polymorphic function $g \colon \forall \alpha.\, (\alpha \to \alpha) \to (\alpha \to \alpha)$ is not even the right *kind* of thing to be a natural transformation from the Hom functor to itself: g is only indexed by *one* type variable α, whereas Hom is a *difunctor*, a functor taking in one covariant and one contravariant argument. The notion of a *dinatural transformation* [ML78, Chapter IX] does not provide a general solution either, since dinaturals do not compose. Instead, the analogy to naturality is left there; the approach taken by Reynolds [Rey83], and consequently by most of the literature on parametric polymorphism, is to state parametricity in terms of *relations* instead of *functions*. Indeed, [HRR14] goes so far as to suggest that Reynolds's solution—"to generalize homomorphisms from functions to relations"—ought to be carried out across mathematics more broadly.

The present author sought to push back on this suggestion, on account of the more cumbersome nature of relational calculi, as well as a desire not to reinvent category theory in a relational mould (e.g. the theory of allegories [FS90]). However, to meet this challenge, defenders of function-based mathematics would need to formulate parametricity in a functional, category-theoretic way, i.e. extend the notion of naturality to mixed-variant functors so as to complete the connection above. In the preprint *Paranatural Category Theory* [Neu23], I sought to develop the category theory surrounding the most promising candidate—*strong dinatural transformations* [Mul92]—towards such a possible solution. The purpose of this talk is discuss the current status of this theory, and the difficulties that have emerged since the first draft of the preprint.

One area of focus will be the failure of the *di-Yoneda Lemma*—an analogue of the Yoneda Lemma, for difunctors and strong dinatural transformations—as originally stated. If the category of difunctors and strong dinatural transformations had such a Yoneda Lemma, then it would be possible to define exponentials in this category analogously to how they're defined in presheaf categories, thereby (potentially) bypassing some of the known issues with using strong dinatural transformations as a formalism for parametricity. I speculate on whether some restricted class of difunctors can be identified which *do* have such a Yoneda Lemma, and whether this class includes the examples important in practice.

I'll also cover some progress towards building a strong (co)end calculus. The original preprint included a development of this calculus, which generalizes the work of Awodey, et al. [AFS18]

[1] See the summary in [HRR14, Sect. 1] and the references cited there.

encoding inductive types, to cover coinductive and existential types; it also included the Yoneda-like Lemmas due to Uustalu [Uus10], connecting strong dinaturality to initial algebras/terminal coalgebras. However, the original preprint left much of the connection existing work on the (co)end calculus unexplored and several questions unanswered, which I hope to address in this talk.

References

[AFS18] Steve Awodey, Jonas Frey, and Sam Speight. Impredicative encodings of (higher) inductive types. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 76–85, 2018.

[FS90] Peter J Freyd and Andre Scedrov. *Categories, allegories*. Elsevier, 1990.

[HRR14] Claudio Hermida, Uday S Reddy, and Edmund P Robinson. Logical relations and parametricity–a reynolds programme for category theory and programming languages. *Electronic Notes in Theoretical Computer Science*, 303:149–180, 2014.

[ML78] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer New York, 1978.

[Mul92] Philip S Mulry. Strong monads, algebras and fixed points. *Applications of Categories in Computer Science*, 177:202–216, 1992.

[Neu23] Jacob Neumann. Paranatural category theory. https://arxiv.org/abs/2307.09289, 2023.

[Rey83] John C Reynolds. Types, abstraction and parametric polymorphism. In *Information Processing 83, Proceedings of the IFIP 9th World Computer Congres*, pages 513–523, 1983.

[Uus10] Tarmo Uustalu. A note on strong dinaturality, initial algebras and uniform parameterized fixpoint operators. In *FICS*, pages 77–82, 2010.

[Wad89] Philip Wadler. Theorems for free! In *Proceedings of the fourth international conference on Functional programming languages and computer architecture*, pages 347–359, 1989.