A sampling of synthetic 1-category theory

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Recent years have seen the gradual emergence of *directed type theory* [LH11, Nuy15, RS17, Nor19, WL20], a variant of Martin-Löf type theory (or, more ambitiously, of *homotopy type theory*) which has *directed paths* (Hom-types) instead of *invertible paths* (identity types). This development has brought into focus a remarkable theory, connecting the *directed J-rule(s)*, the *directed univalence axiom*, the famous *parametricity* result [Rey83] from the theory of functional programming languages, and the *Yoneda Lemma*. My aim in this talk is to chart the major landmarks of this theory, and motivate further exploration of it.

Directed Type Theory and Synthetic Category Theory. The variety of directed type theory practiced here is the "directed analogue" of the theory of Hofmann and Streicher's *groupoid model* [HS95]; indeed, this is the theory of the *category model.* [1] The theory of the groupoid model provides a language for *synthetic (1-)groupoids*, and this theory will serve as a theory of synthetic 1-*categories*: we can understand the types as categories, the terms as objects, and the directed paths as morphisms—the lack of symmetry for directed paths corresponds to the fact that morphisms are not, in general, isomorphisms. Accordingly, we will have to pay careful attention to *polarities*, that is, *co-* and *contra-variance*: for each type (category) A, there will be another type (category) A^- with the same terms (objects) but with all the directed paths (morphisms) reversed. Throughout, we will have to carefully 'tag' various terms as contravariant by having them be of a negated type. For instance, in the formation of the type $\mathsf{Hom}_A(t, t')$ of directed paths from t to t', we will need t to be a term of A^- and t' a term of A.

Directed Path Induction. As is typical with category theory, many of the key constructs in this directed type theory will come in dual pairs. For instance, we have two ways of typing `refl`:

$$\frac{t\colon A^-}{\mathtt{refl}^+_t\colon \mathsf{Hom}_A(t,-t)} \qquad \frac{t'\colon A}{\mathtt{refl}^-_{t'}\colon \mathsf{Hom}_A(-t',t')}$$

where putting a minus sign before a term means coercing from A^- to A, or vice-versa (recall that A and A^- have the same objects, that is, the same terms). Notice it is only *necessary* to state one of these, and prove the other as a consequence. Similarly, the elimination rule

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[1] A detailed account of this model will shortly be uploaded to the arXiv. The final version of this abstract will include a reference to it.

for Hom-types, the directed J rules, can also be stated as a dual pair.

$$\frac{\begin{array}{c} t \colon A^- \\ z \colon A, p \colon \mathsf{Hom}_A(t,z) \vdash M \;\mathsf{type} \\ m \colon M[-t, \mathtt{refl}_t^+] \\ t' \colon A \\ p \colon \mathsf{Hom}_A(t,t') \end{array}}{J_M^+ \; m \; t' \; p \colon M[t',p]} \qquad \frac{\begin{array}{c} t' \colon A \\ z \colon A^-, p \colon \mathsf{Hom}_A(x,t') \vdash M \;\mathsf{type} \\ m \colon M[-t', \mathtt{refl}_{t'}^-] \\ t \colon A \\ p \colon \mathsf{Hom}_A(t,t') \end{array}}{J_M^- \; m \; t \; p \colon M[t,p]}$$

Semantically, these are consequences of the fact that identity morphisms are initial in the coslice category and terminal in the slice category, respectively. With the J-rule(s) in hand, we're able to get started with doing synthetic category theory in the language of directed type theory, e.g. by constructing the composition of morphisms, proving associativity and identity laws, constructing the action of functors on morphisms, and so on.

Directed Univalence. In this setting, we can also introduce a directed universe Set of sets and investigate the nature of directed paths $F \colon \mathsf{Hom}_{\mathsf{Set}}(A,B)$. Again analogously to the groupoid model, the category model of validates a 1-truncated, directed version of the univalence axiom, expressing that such directed paths are the same thing as functions $A \to B$. As with ordinary (undirected) homotopy type theory and its univalence axiom, there appear to be three ways to introduce directed univalence: it could be assumed as an *axiom* (like how (undirected) univalence is in Book HoTT); it could be proved as a *theorem* in a richer theory based on the directed interval (analogous to treatments of univalence in simplicial or cubical HoTT); or perhaps made into a *definition* as part of a directed higher observational type theory [Shu22, AKS22]. The second approach is already well-developed;[2] this investigation is intended as groundwork towards the latter approach.

Parametricity. This directed 1-type theory also provides a suitable framework to achieve Altenkirch and Sestini's [Ses19, Alt19] proposal to internalize parametricity using directed path induction and directed univalence. Consider functions $F, G \colon A \to B$ in this theory, which we understand as functors from the synthetic category A to the synthetic category B. Given a "transformation"

$$\alpha \colon \prod_{a:A} \mathsf{Hom}_B(F\,a, G\,a)$$

we can prove that such a transformation must be *natural* in the appropriate sense. This can be easily proved by directed path induction, regardless of how F, G, and α are defined. We get naturality just in virtue of the type of α, that is to say, *for free* (in the sense of [Wad89]). In the case where A and B are Set, we can use directed univalence to reason that α is natural with respect to the usual function composition of functions, matching the more familiar notion of 'natural'.

Dependent Yoneda. Finally, this setting also allows us to observe a striking correspondence, noted by Riehl and Shulman, between directed path induction and a dependent form of the Yoneda Lemma. Riehl and Shulman's more powerful and elaborate theory allows them to make this connection in the ∞-category setting, but it is worth noting that the more elementary framework employed here still manages to achieve the same result for the variety of category theory it can capture synthetically, 1-category theory.

[2] Two such theories are the simplicial approach pursued by Riehl and Shulman[RS17], and the cubical approach by Licata and Weaver[WL20]. Both now being developed as computer proof assistants, the `rzk` language and a `cooltt` extension, respectively. See also [RCS18] for a discussion of directed univalence.

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