Towards Directed Higher Observational Type Theory

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An essential lesson of Martin-Löf Type Theory is that *identity is a groupoid*: if $x = y$ is a type of data, then the familiar properties of reflexivity, symmetry, and transitivity of equality become groupoidal operations (identity, inverse, and composition, respectively) on that data. In this way, the identity types of MLTT become a language for reasoning synthetically about groupoids. The connection between groupoids and MLTT is further strengthened by the groupoid model of type theory[1], where groupoids are used to give semantics for MLTT. This leads to a natural question: what if we used categories instead of groupoids? Would that furnish us with a language for synthetic *category theory*?

This question is taken up by the branch of type theory known as **directed type theory**. Directed type theory pays attention to the variance of terms, having different types for terms which appear positively (covariantly), negatively (contravariantly), or both. As such, it is a natural setting for synthetic category theory: instead of identity types – which are inherently symmetrical – directed type theory has Hom types, which are inherently directed, or asymmetrical. In recent years, there have been several attempts to synthesize directed type theory with homotopy type theory (HoTT). [3, 4, 5]

Another important (and very recent) project in homotopy type theory is the development of so-called **higher observational type theory**, or HOTT.[1] This type theory seeks to combine the advantages of "Book HoTT" and cubical HoTT, and avoid some perceived disadvantages of either. In particular, HOTT seeks to make "observational" equivalences in HoTT, e.g.

$$(a, b) =_{A\times B} (a', b') \quad \simeq \quad (a =_A a') \times (b =_B b')$$

into bona-fide judgmental equalities. This also extends to the Univalence Axiom, which HOTT makes true "by definition".

In this talk, I lay some of the groundwork for a confluence of these studies: directed higher observational type theory. The investigation into this idea is in its early stages, and many of the results here will be conjectural. But, although many of the fine details remain unclear, this theory promises to give an account of directed HoTT and synthetic category theory which benefits from the advantages of higher observational type theory. I focus on two tasks: building appropriate semantic notions for directed HOTT, and trying out synthetic category theory in the resulting syntax.

1 The Category Model and Directed CWFs

The semantic notion that I define is that of a *directed category with families* – which, to my knowledge, has not been explicitly articulated as of yet. Hofmann and Streicher define the groupoid model of type theory as a CWF, and use groupoids to satisfy the key constructs. But simply replacing groupoids with categories in their definition again yields a CWF – the *category model*. This model supports operations (such as negative types and contravariant

[1] To my knowledge, there has not been published work on HOTT. The characterization of HOTT contained here is based on Michael Shulman's 2022 talks (which are linked at [2]), Ambrus Kaposi's presentation at TYPES 2022, and my conversations with Thorsten Altenkirch.

context extension) which are not available in ordinary CWFs. Our definition of directed CWF will be an attempt to abstractly specify all these features.

Analogously to ordinary CWFs, directed CWFs may be equipped with further type constructors. Our main focus will be the directed analogue of identity types: Hom types.

$$\frac{M\colon \mathsf{Tm}(\Gamma, A^-) \quad N\colon \mathsf{Tm}(\Gamma, A)}{M \Rightarrow N\colon \mathsf{Ty}\,\Gamma} \qquad \frac{M\colon \mathsf{Tm}(\Gamma, A^0)}{\mathsf{refl}_M\colon \mathsf{Tm}(\Gamma, M \Rightarrow M)}$$

$$\frac{L\colon \mathsf{Tm}(\Gamma, A^-) \quad M\colon \mathsf{Tm}(\Gamma, A^0) \quad N\colon \mathsf{Tm}(\Gamma, A) \quad F\colon \mathsf{Tm}(\Gamma, L \Rightarrow M) \quad G\colon \mathsf{Tm}(\Gamma, M \Rightarrow N)}{G \circ F\colon \mathsf{Tm}(\Gamma, L \Rightarrow N)}$$

Note that we must explicitly annotate our types with "polarities": either covariant (A), contravariant (A^-), or bivariant (A^0) – the semantics of these operations relies critically on these variances being correct. With these definitions, we can do some simple synthetic 1-category theory in the syntax of directed type theory, and our semantic definition suggests extensions to higher category theory.

2 Synthetic Category Theory in Telescopic HOAS

Informed by these semantic investigations, I propose a telescopic higher-order abstract syntax for directed higher observational type theory. This presentation will match undirected HOTT as closely as possible: we work in a presheaf setting, and make telescopic judgments $\Delta \vdash \mathcal{J}$ relative to a given context Γ. Like the identity types of undirected HOTT, the Hom-types of directed HOTT will be defined homogeneously for telescopes and heterogeneously on types, and will be given computational import via a general congruence term which computes on term-formers.

As a paradigm example, we will consider synthetic slice categories: for A a type and $M\colon A$, define

$$A/M :\equiv \sum_{N:A} \mathsf{Hom}_A(N, M).$$

The Hom-types between terms of type A/M will be determined analogously to identity types in undirected HOTT, but taking into account the contravariance of the type family $z : A^- \vdash \mathsf{Hom}_A(z, M)$. To do this, we will mimic the *definitional univalence* of undirected HOTT and define Hom-types in the universe by functional relations. Combining this with an appropriate notion of *contravariant* heterogeneous/dependent Hom-types, we will find that our synthetic slice categories have exactly the Hom-types we expect.

3 Further Questions

There are several open issues, which will hopefully be illuminated by further study:

1. How can we make precise the directed presheaf setting? How do dependent types and universes operate in this setting?
2. How can we best equip our Hom types with their own identity types, in order to be able to make equations and commutative diagrams in synthetic category theory?
3. What changes are necessary to encode higher category theoretic structure?
4. What is the precise relation of this theory to existing directed type theories, specifically to existing notions of directed HoTT?

References

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